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A DIRECT METHOD OF CALCULATING BOTTOM ORBITAL VELOCITY UNDER WAVES

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ABSTRACT

A method is presented for calculating the bottom orbital velocity under a wave simply and directly from its known height and period, and the water depth. When suitably nondimensionalised the results all fall on a single curve, with separate curves for monochromatic and random (JONSWAP spectrum) waves. The r.m.s velocity under random waves may be smaller or larger than that produced by a monochromatic wave of height H_s and period T_z, depending on the water-depth. Direct methods of obtaining the effective period of the bottom velocity under random waves are also presented; these periods can be appreciably longer than T_z.

CONTENTS

1 INTRODUCTION

In many aspects of coastal engineering and oceanography it is necessary to know the orbital velocity at the sea bed produced by surface waves. Applications include sediment transport problems, forces on pipe-lines and structures at the sea bed, and the dissipation of wave energy. Frequently the bottom orbital velocity has to be deduced from surface measurements of wave height and period.

For monochromatic waves the appropriate quantity is the maximum bottom orbital velocity U_m during the wave cycle. However, a naturally occurring random sea will have a broad spectrum of frequencies. Generally, information on the waves will be given in terms of the significant wave-height H_s and the zero-crossing period T_z (or the peak period T_p). It is tempting to assume that the sea can be represented by a monochromatic wave of height H_s and period T_z (or T_p). However, this may not be a good approximation as the attenuation of orbital velocity with depth depends strongly on wave period, so that the dominant waves at the bottom will have a period different to either T_z or T_p. The near-bottom velocity cannot now be described by a single U_m, and it is usual to describe it by the standard deviation U_{rms} of the time-series of instantaneous velocities. In some applications it is important to know the effective period of the orbital velocity, as well as the velocity itself.

The calculation of U for monochromatic waves is not straightforward because it is necessary to solve the dispersion relation for the wave-number, which must be done graphically, iteratively, or as a series approximation. Calculation of U_{rms} from a given surface elevation spectrum is considerably more laborious. The usual procedure is to convert the elevation spectrum to a bottom-velocity spectrum , which involves solving the dispersion relation at each frequency, and then integrating the resulting spectrum

over the frequency range to yield U_{rms}^2. Calculation of the effective period is equally laborious.

The purpose of this report is to present a method of calculating U_m, U_{rms}, and the effective period, directly from the known quantities H_s and T_z together with the water depth. The results in each case are presented as single curves which are given in three alternative forms : graphically; as tabulated values; and as explicit algebraic expressions which approximate the curves closely.

2 MONOCHROMATIC WAVES

Consider a wave of amplitude $a = H/2$, and radian frequency $\omega = 2\pi/T$, where H and T are the wave height and period respectively, which gives rise to a maximum orbital velocity U_m at the sea-bed (or, more correctly, just outside the thin wave boundary layer near the bed). Then U_m is obtained using small-amplitude linear wave theory from

$$\frac{U_m}{a} = \frac{\omega}{\sinh (kh)} \qquad (1)$$

The wavenumber k is related to the frequency ω by the dispersion relation

$$\omega^2 = gk \tanh (kh), \qquad (2)$$

where g is the acceleration due to gravity and h is the water depth. Define dimensionless variables:

$$x = \frac{\omega^2 h}{g} \qquad (3)$$

$$y = kh \qquad (4)$$

$$F_m = \frac{U_m^2 h}{a^2 g} \tag{5}$$

Then Equation (1) becomes, after use of Equation (2),

$$F_m = \frac{2y}{\sinh(2y)} \tag{6}$$

and the dispersion relation, Equation (2), becomes

$$x = y \tanh y. \tag{7}$$

The dimensionless transfer function F_m cannot be written explicitly in terms of x, and hence in terms of H and T, because the dispersion relation, Equation (7), cannot be written explicitly as y(x). However, as Equation (7) gives a one-to-one correspondence between x and y, we see from Equation (6) that F_m is a parametric function of x alone. Both F_m and x contain only the known quantities H, T, h and g, and the required quantity U_m. Thus a plot of F_m versus x (obtained by using y as a parameter in Equations (6) and (7)) allows U_m to be obtained directly from the known quantities (Fig 1). For small values of x (shallow-water waves) the value of F_m tends to one, and F_m decreases monotonically with x until it becomes very small for $x > 4$ (deep-water waves).

The quantities F_m and x are unnecessarily complicated for practical calculations, as they contain the squares of the quantities of interest and also contain some unnecessary constants. We therefore define more readily usable quantities by first introducing the natural scaling period T_n defined by

$$T_n = \left(\frac{h}{g}\right)^{\frac{1}{2}} \tag{8}$$

Then the required dimensionless quantities are

$$\frac{U_m T_n}{2H} \equiv \frac{F_m^{\frac{1}{2}}}{4} \tag{9}$$

and

$$\frac{T_n}{T} \equiv \frac{x^{\frac{1}{2}}}{2\pi} \tag{10}$$

A plot of $U_m T_n/2H$ versus T_n/T (Fig 2) can be used directly for obtaining U_m from H, T, g and h. For computer application, values of $U_m T_n/2H$ are tabulated against T_n/T in Table 1.

A 3-part explicit algebraic expression can be found which fits the curve in Figure 2 closely, as follows:

$$\begin{aligned} F_m^{\frac{1}{2}} \equiv \frac{2U_m}{H}\left(\frac{h}{g}\right)^{\frac{1}{2}} &= (1 - 0.670x + 0.110\,x^2)^{\frac{1}{2}}, \quad 0 \leq x < 1 \\ &= 1.72\,x^{\frac{1}{2}}\,e^{-0.9529x} \quad , \quad 1 \leq x < 3.2 \\ &= 2x^{\frac{1}{2}}\,e^{-x} \quad , \quad 3.2 \leq x < \infty \end{aligned} \tag{11}$$

with

$$x = \left(\frac{2\pi}{T}\right)^2 \frac{h}{g}.$$

Equation (11) fits the exact curve in Figure 2 to an accuracy of better than ±1% over the entire range $0 \leq x < \infty$. The first and third parts of Equation (11) are based respectively on small and large argument approximations to sinh y and tanh y in Equations (6) and (7), together with some optimisation of the coefficients in the first part. Optimisation was also used to determine the coefficients in the middle part. The approximation given by Equation (11) has not been shown on Figure 2, because it is indistinguishable from the exact curve.

3 A RANDOM SEA

Under natural conditions the wave climate is represented by a spectrum of waves of different frequencies, amplitudes and directions. In many cases the only parameters which are known about the sea-conditions are the significant wave height H_s and the zero-crossing period T_z. The best that can then be done is to fit a realistic surface elevation spectrum $S_\eta(\omega)$ to these two parameters. One of the most widely accepted two-parameter spectra is the JONSWAP spectrum (Hasselman et al, 1973), given by

$$S_\eta(\omega) = 2\pi\alpha\, g^2\omega^{-5} \exp\left\{-\frac{5}{4}\left(\frac{\omega}{\omega_p}\right)^{-4}\right\} \gamma^{\phi(\omega)}$$

where (12)

$$\phi(\omega) = \exp\left\{-\frac{(\omega-\omega_p)^2}{2\beta^2\omega_p^2}\right\}.$$

Here ω_p is the radian frequency at the peak of the spectrum, γ and β are constants, and α is a variable which depends on the wind-speed and duration. We use the standard values of the constants, $\gamma = 3.3$ and $\beta = 0.07$ for $\omega > \omega_p$, $\beta = 0.09$ for $\omega < \omega_p$. The variables α and ω_p can be related to H_s and T_z respectively, so that a particular sea-state described only by H_s and T_z corresponds to a particular JONSWAP spectrum.

An additional complication of a random sea is that there is an appreciable spread in the wave directions, which is generally expressed by multiplying Equation (12) by a spreading function. For calculations of the wave energy dissipation rate the form of the spreading function can influence the dissipation rate by up to 20% (Brampton et al, 1984). However, because the bottom orbital velocity is related linearly to the surface elevation η, it is seen that the relationship between the quantities U_{rms}^2 and H_s^2 ($=16\,\sigma_\eta^2$) is independent of the spreading function.

The bottom velocity spectrum $S_u(\omega)$ is obtained by applying the dimensional transfer function given by Equation (1) to each frequency in the elevation spectrum:

$$S_u(\omega) = \frac{\omega^2}{\sinh^2(kh)} \; S_\eta(\omega) \tag{13}$$

The variance of the bottom velocity is then obtained by integrating $S_u(\omega)$ over frequency

$$U^2_{rms} = \int_0^\infty S_u(\omega)\, d\omega \tag{14}$$

We now define dimensionless variables analogous to those for monochromatic waves (Equations 3 and 5):

$$x_z = \left(\frac{2\pi}{T_z}\right)^2 \frac{h}{g} \; ; \quad F_r = \left(\frac{4U_{rms}}{H_s}\right)^2 \left(\frac{h}{g}\right) \tag{15}$$

We note here that the standard deviation of the surface elevation of a random sea is $H_s/4$, and the standard deviations of the surface elevation and bottom velocity of a monochromatic wave are $H/\sqrt{2}$ and $U_m/\sqrt{2}$ respectively. Thus, to make analogous quantities for monochromatic and random waves correspond in meaning, we have introduced the factor 4 into the definition of F_r, and the factor 2 into the quantity $U_m T_n/2H$ defined earlier (see Appendix).

If we now further define $x_p = \omega_p^2 h/g$, and write $\sinh^{-2} y = \text{func}_1(x)$ via Equation (7), then Equation (14) becomes

$$\begin{aligned} U^2_{rms} &= \alpha\, gh \int_0^\infty x^{-3/2} \exp\left\{-\frac{5}{4}\left(\frac{x}{x_p}\right)^{-2}\right\} \gamma^{\phi\left(\frac{x}{x_p}\right)} \text{func}_1(x)\, dx^{\frac{1}{2}} \\ &= \alpha\, gh\, \text{func}_2\,(x_p) \end{aligned} \tag{16}$$

Also, from the definition of H_s in terms of the zeroth moment of the spectrum Equation (12), we have

$$\left(\frac{H_s}{4}\right)^2 = \int_0^\infty S_\eta(\omega)\, d\omega$$

$$= \alpha h^2 \int_0^\infty x^{-5/2} \exp\left\{-\frac{5}{2}\left(\frac{x}{x_p}\right)^{-2}\right\} \gamma^{\phi\left(\frac{x}{x_p}\right)} dx^{\frac{1}{2}}$$

$$= \alpha\, h^2\, \mathrm{func}_3\,(x_p) \qquad (17)$$

From Equations (16) and (17) we obtain an expression for the dimensionless transfer function F_r given by Equation (15):

$$F_r = \frac{\mathrm{func}_2\,(x_p)}{\mathrm{func}_3\,(x_p)} \qquad (18)$$

Thus F_r is a function of x_p alone. By expressing T_z^2 as the ratio of the second and zeroth moments of $S_\eta(f)$ it can be related to the peak period T_p of the JONSWAP spectrum given by Equation (12). With the values of β and γ given earlier,

$$T_p \equiv \frac{2\pi}{\omega_p} = 1.281\ T_z \qquad (19)$$

Thus x_z is proportional to x_p, and it follows that F_r is a function only of x_z.

Values of F_r for a range of values of x_z have been calculated by performing numerically the integration given by Equation (14). An adaptation of a more general existing computer program described by Brampton et al (1984) was used, with an integration step of 0.1s and limits of the integration taken between periods of 0.1s and $5T_p$. The resolution and limits are ample to give good accuracy. The resulting curve (Fig 1) follows the curve for monochromatic

waves for small values of x_z, but becomes increasingly larger than it as x_z increases.

For simpler graphical use we have plotted $U_{rms}T_n/H_s$ versus T_n/T_z (Fig 2) where T_n is defined by Equation (8). Values of $U_{rms}T_n/H_s$ are tabulated against T_n/T_z in Table 1. For the random wave case it is not straightforward to obtain asymptotic expressions for F_r for small and large x_z, as was done for monochromatic waves. Instead we have employed curve-fitting techniques to obtain an explicit algebraic expression which fits the curve in Figure 2 closely:

$$\frac{U_{rms}T_n}{H_s} = \frac{0.25}{(1 + At^2)^3}$$

where

$$A = \left[6500 + (0.56 + 15.54t)^6\right]^{1/6} \qquad (20)$$

and

$$t = \frac{T_n}{T_z} = \frac{1}{T_z}\left(\frac{h}{g}\right)^{\frac{1}{2}}$$

Equation (20) fits the JONSWAP curve in Figure 2 to an accuracy of better than 1% in the range $0 < t < 0.55$. Again we have not plotted Equation (20) on Figure 2 because it is indistinguishable from the exact curve.

We have calculated F_r only up to $x_z = 11.8$, ie, $T_n/T_z = 0.55$, because for larger values of x_z the bottom velocity is very small. For $T_n/T_z = 0.55$, Figure 2 gives $U_{rms}T_n/H_s = 0.0038$, so that

$$U_{rms} = 0.0069\,\frac{H_s}{T_z} \quad \text{at} \quad T_n/T_z = 0.55 \qquad (21)$$

This provides an upper bound to velocities for $T_n/T_z >$ 0.55. For example, if $H_s = 4$m, $T_z = 4$s, $h = 47.5$m,

$g = 9.81 ms^{-2}$, then $T_n/T_z = 0.55$ and $U_{rms} = 0.0069 ms^{-1}$.

4 PERIOD OF BOTTOM ORBITAL VELOCITY

Although it is clear from the frequency dependence of the transfer function that the effective period of the bottom velocity will be larger than that of the wave elevations, it is less clear how the "effective period" should be defined. We examine here the period corresponding to the peak in the spectrum of bottom orbital velocity. This makes the largest contribution to the variance U_{rms}^2, which is the form in which wave-effects often appear in applications to sediment transport or to forces on structures near the sea-bed.

We therefore wish to compare the period T_{pu} at the peak of the velocity spectrum $S_u(\omega)$ with the period $T_p \equiv 2\pi/\omega_p$ at the peak of the elevation spectrum $S_\eta(\omega)$. The maximum of $S_u(\omega)$ is found by expressing Equation (13) in terms of F_m from Equations (1) and (5), and differentiating with respect to ω. After setting $dS_u/d\omega = 0$ and dividing through by S_u we obtain:

$$\frac{1}{F_m}\frac{dF_m}{d\omega} = -\frac{1}{S_\eta}\frac{dS_\eta}{d\omega} \qquad (22)$$

Making the substitution $\phi = \omega/\omega_p$, Equation (22) becomes

$$\frac{1}{F_m}\frac{dF_m}{dy}\frac{dy}{dx}\frac{dx}{d\omega} = -\frac{1}{S_\eta}\frac{dS_\eta}{d\phi}\frac{d\phi}{d\omega} \qquad (23)$$

Substitution of F_m and dF_m/dy from Equation (6), dy/dx from Equation (7), $dx/d\omega$ from Equation (3), S_η and $dS_\eta/d\phi$ from Equation (12), and $d\phi/d\omega = 1/\omega_p$, enables Equation (23) to be written in the form

$$P(x) = Q(\phi) \quad (24)$$

where

$$P(x) = \frac{(2+4y)\ e^{-4y} + 4y - 2}{1-e^{-4y} + 4y\ e^{-2y}}$$

and

$$Q(\phi) = 5\phi^{-4} - 5 + \frac{\ln\gamma}{\beta^2}\ \phi\ (1-\phi)\ \exp\ \left\{-\frac{1}{2\beta^2}\ (1-\phi)^2\right\}$$

The functions $Q(\phi)$ vs ϕ and $P(x)$ vs x (using y as parameter) can be plotted. Then if we denote by x_1 and ϕ_1 the values of x and ϕ which correspond to equal values of $P(x)$ and $Q(\phi)$, thereby satisfying Equation (24), we obtain, using the definition of ϕ,

$$\frac{T_{pu}}{T_p} = \left(\frac{x_p}{x}\right)^{\frac{1}{2}} = \frac{1}{\phi_1}$$

and (25)

$$x_p = \frac{x_1}{\phi_1^2}$$

By picking off values of $P(x) = Q(\phi)$, and making use of Equation (19), a plot of T_{pu}/T_z can be constructed (Fig 3). For small values of T_n/T_z (shallow-water waves) we find that T_{pu} tends to $T_p \equiv 1.281\ T_z$. As T_n/T_z increases, T_{pu}/T_z increases first slowly, then rather rapidly close to $T_n/T_z = 0.4$, and finally slowly again for $T_n/T_z > 0.5$, at which point T_{pu} is in excess of $1.7\ T_z$.

Because the curve of T_{pu}/T_z vs T_n/T_z is not a simple shape we have not attempted to fit an algebraic approximation to it. Values of T_{pu}/T_z are tabulated against T_n/T_z in Table 1.

5 EXAMPLES

As illustrations consider a monochromatic wave of height $H = 5m$, period $T = 8s$, for two water-depths $h = 10m$ and $50m$.

For $h = 10m$, Eq (8) gives $T_n = 1.02s$, and hence $T_n/T = 0.127$. From the "monochromatic" curve in Fig 2 we obtain $U_m T_n/2H = 0.196$, and thus $U_m = 1.92ms^{-1}$.

For $h = 50m$ the corresponding values are $T_n = 2.26s$, $T_n/T = 0.282$, $U_m T_n/2H = 0.038$, and $U_m = 0.168ms^{-1}$.

Now consider a random sea having a JONSWAP spectrum with $H_s = 5m$ and $T_z = 8s$, in the same water-depths.

Then for $h = 10m$, Eq (8) gives $T_n = 1.02s$, and $T_n/T_z = 0.204$, and thus $U_{rms} = 1.00ms^{-1}$.

For $h = 50m$ the corresponding values are $T_n = 2.26s$, $T_n/T_z = 0.282$, $U_{rms}\ T_n/H_s = 0.087$, and $U_{rms} = 0.192ms^{-1}$.

In order to compare the random sea with a monochromatic wave of height H_s and period T_z it is first necessary to convert the velocity amplitude of the monochromatic wave to the corresponding root-mean-square value by $U_{rms} = U_m/\sqrt{2}$. Then for $h = 10m$ the random sea value $U_{rms} = 1.00ms^{-1}$ compares with the monochromatic wave value $U_{rms} = 1.36ms^{-1}$. By contrast, for $h = 50m$ the random sea value $U_{rms} = 0.192ms^{-1}$ compares with the monochromatic wave value $U_{rms} = 0.119ms^{-1}$. Thus in shallow water an estimate of bottom orbital velocity based on a monochromatic wave of height H_s and period T_z will be a serious overestimate, but in deep water it will be a serious underestimate. The cross-over point, at which the monochromatic and random waves give the same U_{rms}, occurs at a depth given (using Fig 2) by $h = 0.049g\ T_z^{\ 2}$.

The peak-period of the bottom orbital velocity spectrum for the random sea with h = 10m, is obtained from Fig 3 with $T_n/T_z = 0.127$, for which $T_{pu}/T_z = 1.296$ giving $T_{pu} = 10.4$s. The corresponding value for h = 50m is 10.6s. Neither value is very different from the peak-period of the surface elevation spectrum given by Eq (19) as $T_p = 10.2$s. For a JONSWAP spectrum, which is relatively strongly peaked, T_{pu} will be appreciably different from T_p only for rather short period waves or rather deep water.

6 SUMMARY

Methods have been presented for calculating directly the bottom orbital-velocity and effective bottom period of waves of known height and period in water of depth h. The results are presented graphically (for visual use), as tables (for computer application by look-up table), and as explicit algebraic expressions accurate to ±1% (for use on pocket calculators and micro-computers). The methods can be summarised as follows:

1. Results, both for monochromatic and random waves, are scaled by the natural scale period $T_n = (h/g)^{\frac{1}{2}}$.

2. For a monochromatic wave of height H and period T the amplitude U_m of the bottom orbital velocity can be obtained from the plot of $U_m T_n/2H$ versus T_n/T given in Figure 2, or from Table 1 or Equation (11). These results can be used for laboratory as well as prototype waves.

3. For a random sea characterised by the significant wave height H_s and zero-crossing period T_z, the root-mean-square bottom orbital velocity U_{rms} can be obtained from the plot of $U_{rms} T_n/H_s$ versus T_n/T_z given in Figure 2 or from Table 1 or Equation (20). This is based on a JONSWAP form for the elevation spectrum. Results are presented

for $0 < T_n/T_z < 0.55$ which covers the entire range of practical interest, but for values outside this range an upper bound to U_{rms} is given by Equation (21).

4. The rms bottom orbital velocity calculated by assuming a JONSWAP spectrum (item 3 above) is larger or smaller than that calculated by assuming a monochromatic wave of height H_s and period T_z (item 2 above) depending on whether h is larger or smaller than $0.049g\ T_z^{\ 2}$ respectively. The difference may exceed 40% in either case.

5. The period T_{pu} of the peak of the velocity spectrum is larger than the period T_p of the elevation spectrum, by an amount which can be obtained from the plot of T_{pu}/T_z versus T_n/T_z given in Figure 3, or from Table 1. For many purposes the assumption that the effective period at the sea-bed is T_p is adequate.

6. Where necessary, use the relation $T_p = 1.281\ T_z$ throughout.

7 ACKNOWLEDGEMENTS

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8 REFERENCES

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2. Hasselman K et al (1973). Measurements of wind-wave growth and swell decay during the Joint North Sea Wave Project (JONSWAP). Deutsche Hydrographische Zeitschrift, Supplement A8, No 12.

Table

Table 1: Values of non-dimensional bottom orbital velocity for monochromatic and random (JONSWAP spectrum) waves, and peak-period of bottom orbital - velocity spectrum.

$T_n = \sqrt{\left(\frac{h}{g}\right)}$

MONOCHROMATIC		JONSWAP SPECTRUM		
T_n/T	$U_m T_n/2H$	T_n/T_z	$U_{rms} T_n/H_s$	T_{pu}/T_z
0.00	0.250	0.00	0.250	1.281
0.02	0.248	0.02	0.248	1.281
0.04	0.244	0.04	0.245	1.282
0.06	0.237	0.06	0.238	1.283
0.08	0.228	0.08	0.230	1.284
0.10	0.216	0.10	0.219	1.285
0.12	0.202	0.12	0.208	1.286
0.14	0.185	0.14	0.196	1.288
0.16	0.165	0.16	0.172	1.290
0.18	0.143	0.18	0.167	1.292
0.20	0.120	0.20	0.150	1.296
0.22	0.097	0.22	0.134	1.299
0.24	0.075	0.24	0.118	1.304
0.26	0.055	0.26	0.103	1.311
0.28	0.039	0.28	0.088	1.319
0.30	0.027	0.30	0.075	1.328
0.32	0.018	0.32	0.063	1.339
0.34	0.012	0.34	0.052	1.353
0.36	0.007	0.36	0.042	1.372
0.38	0.005	0.38	0.033	1.398
0.40	0.003	0.40	0.027	1.448
0.42	0.002	0.42	0.022	1.570
0.44	0.001	0.44	0.017	1.620
0.46	0.000	0.46	0.013	1.653
0.48	0.000	0.48	0.010	1.682
0.50	0.000	0.50	0.008	1.708
0.52	0.000	0.52	0.007	1.731
0.54	0.000	0.54	0.006	1.753

Figures

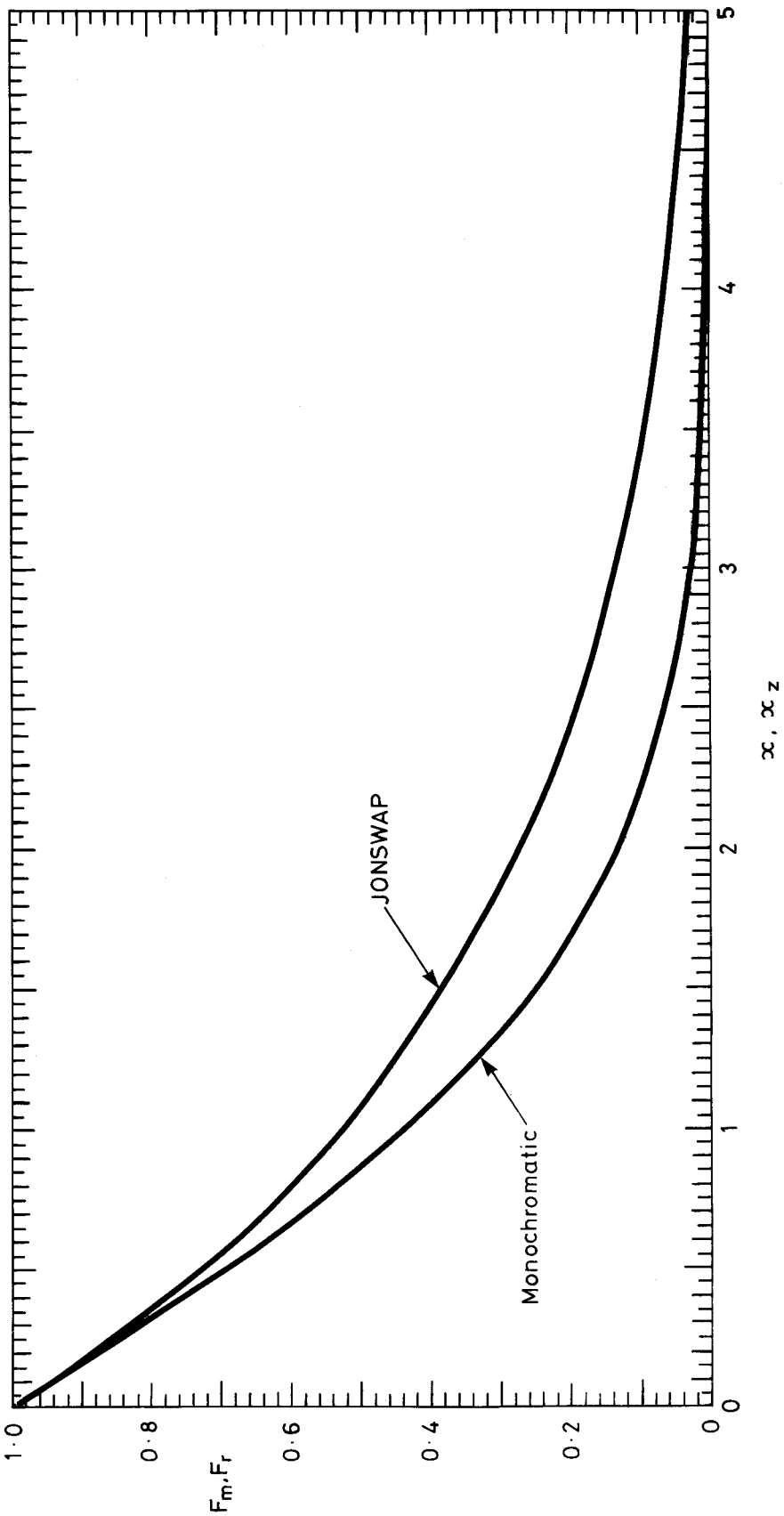

Figure 1 Dimensionless transfer functions as functions of dimensionless frequency. Monochromatic waves : $F_m = U_m^2 h/a^2 g$ versus $x = \omega^2 h/g$. Random waves (JONSWAP spectrum) : $F_r = \left(\frac{4U_{rms}}{H_s}\right)^2 \left(\frac{h}{g}\right)$ versus $x_z = \left(\frac{2\pi}{T_z}\right)^2 \frac{h}{g}$

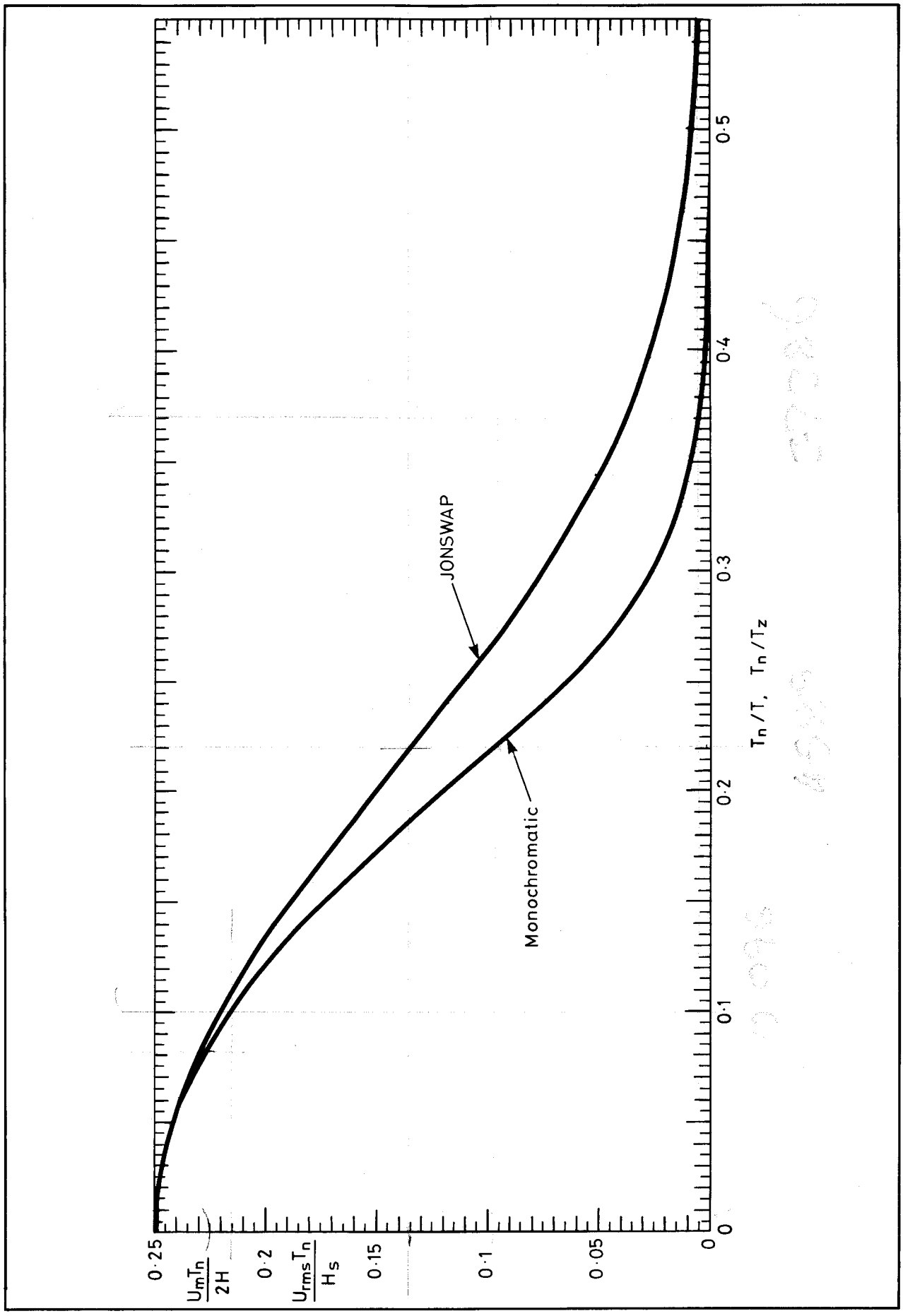

Figure 2 Bottom velocity for monochromatic waves ($U_m T_n/2H$ versus T_n/T) and random waves ($U_{rms}\ T_n/H_s$ versus T_n/T_z), where $T_n = (h/g)^{\frac{1}{2}}$

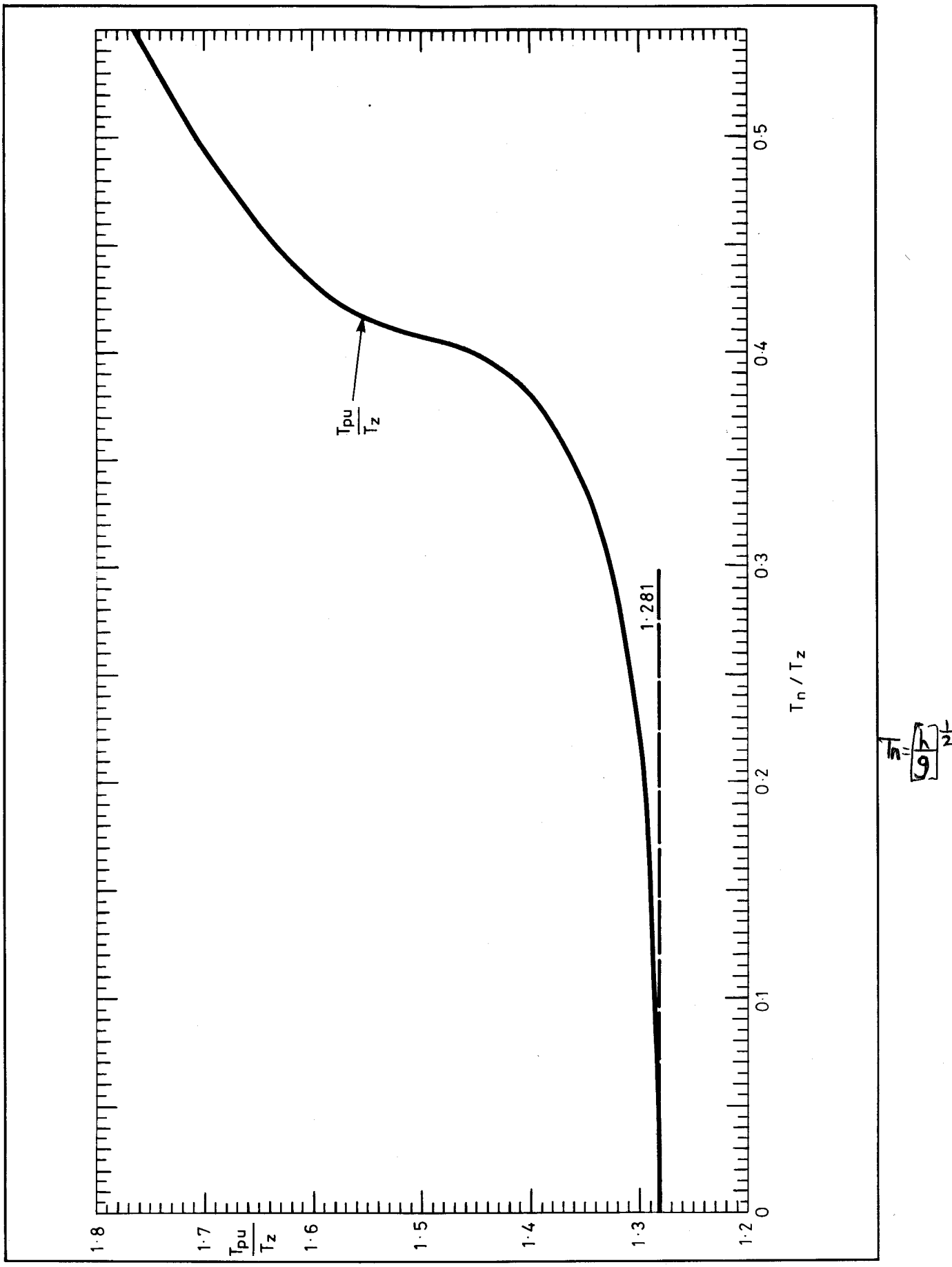

Figure 3 Period T_{pu} of the peak of the bottom orbital velocity spectrum.

Shorter period waves for same h $\frac{T_n}{T_z}$ increases

Appendix

APPENDIX

Relating Monochromatic and Random Wave Parameters

Confusion can easily arise about the interrelationships of the various wave parameters in common use. We clarify here the relationships between the quantities used in this report.

For a sinusoidal monochromatic wave of height H and period T, the angular frequency is:

$$\omega = 2\pi/T, \tag{A1}$$

the amplitude is

$$a = H/2 \tag{A2}$$

and the variation of the surface elevation $\eta(t)$ with time t at a particular point is given by:

$$\eta = a \sin \omega t. \tag{A3}$$

Thus the standard deviation σ_η of the surface is related to H by:

$$H = 2\sqrt{2}\ \sigma_\eta \tag{A4}$$

The bottom orbital velocity U(t) is given by

$$U = U_m \sin \omega t \tag{A5}$$

where U_m is the amplitude of the velocity. The standard deviation σ_u of the velocity, more commonly written as U_{rms}, is thus:

$$\sigma_u \equiv U_{rms} = U_m/\sqrt{2} \tag{A6}$$

For a random sea the significant wave height H_s is defined as:

$$H_s = 4\ \sigma_\eta \tag{A7}$$

where σ_η is now the standard deviation of the random surface elevation $\eta(t)$. (The rms wave height H_{rms} is also sometimes used, and is related to σ_η by $H_{rms} = 2\sqrt{2}\ \sigma_\eta$.) The term "random" is used to distinguish a naturally occurring multi-directional spectrum of waves from a unidirectional monochromatic sinusoidal wave, rather than in the usual statistical sense.

The period can be characterised either by the zero-crossing period T_z or by the angular frequency ω_p at the peak of the surface elevation spectrum, leading to the peak period T_p given by:

$$T_p = 2\pi/\omega_p \tag{A8}$$

For any standard shape of spectrum (JONSWAP, Pierson-Moskowitz, etc) T_p is proportional to T_z, with the constant of proportionality depending on the chosen spectral shape.

The root-mean-square bottom orbital velocity U_{rms} is related to the variance σ_u^2 of the random velocity vector $\underset{\sim}{U}(t)$ by:

See Errata

$$U_{rms}^2 \equiv \sigma_u^2 = \overline{|\underset{\sim}{U}|^2} \tag{A9}$$

– Overbar indicates a mean over a long record.

If a monochromatic wave of height H and a random sea of significant height H_s have the same variance σ_η^2 of the surface elevation, then, using eqs (A4) and (A7), they are related by:

$$H = \frac{H_s}{\sqrt{2}} \tag{A10}$$

If, instead, they have the same variance σ_u^2 of the bottom orbital velocity, then, using eqs (A6) and (A9), they are related by:

$$U_m = \sqrt{2}\ U_{rms} \tag{A11}$$

Thus, using eqs (A2), (A10) and (A11), the random wave quantity defined in eq (15):

$$F_r = \left(\frac{4U_{rms}}{H_s}\right)^2 \left(\frac{h}{g}\right) \tag{A12}$$

corresponds to the monochromatic quantity defined in eq (5):

$$F_m = \frac{U_m^2 h}{a^2 g}.$$

Similarly the monochromatic quantity $U_m T_n/(2H)$ corresponds to the random quantity $U_{rms} T_n/H_s$.