Hydraulics Research
Wallingford

EXTREMES OF SECOND ORDER WAVE QUANTITIES

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Report No SR 161
February 1988

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This report describes work funded by the Department of the Environment under Research Contract PECD 7/6/055 (PIF No HRL 4/85) for which the DoE nominated officer was Dr R P Thorogood. It is published on behalf of the Department of the Environment but any opinions expressed in this report are not necessarily those of the funding Department. The work was carried out in the Maritime Engineering Department of Hydraulics Research, Wallingford, under the Management of Dr S W Huntington.

ABSTRACT

A general theory derived previously for the prediction of extremes of a certain class of non-linear variables, has been shown to be applicable to a general quadratic process. In recent years, a particular quadratic process has been found to be important in harbour design: second order wave effects. Such effects act directly in random seas to produce a long period disturbance known as set-down beneath wave groups which can cause harbour and moored ship resonance. These effects also act indirectly to cause non-linear wave forces on ships which, in turn, contribute to resonant oscillations of vessels on their moorings.

In this report the technique has been applied to two cases of particular interest. One is the prediction of extreme values of set-down in random seas. The other is the prediction of extreme values of paddle movement in physical models of harbours where compensation for set-down is necessary at the wave-maker to avoid the generation of spurious long waves. The prediction of extreme paddle movements with set-down compensation is needed for proper design of the wave generator.

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1 INTRODUCTION

Amongst wave phenomena, those associated with set-down beneath wave groups have become important over the last decade. These are low frequency oscillations coupled to the primary sea state which although small in themselves can, through resonance, excite significant motions in moored vessels (Ref 1) and in enclosed harbours (Ref 2). In laboratory wave generation it has in consequence become important that these effects are correctly reproduced by modifying the signal to the generator (Ref 3). An uncorrected generator will produce spurious low frequency free waves of a similar size to the effects being studied.

Non-linear wave forcing has also been identified as an important aspect in describing the motions of moored ships. As these forces are second order in the wave amplitude their form is not unlike that of set-down, and the signal to a wave generator required to compensate for set-down. After taking the response of the moored vessel into account, the actual movements will also be of a similar form. Thus, the ability to describe the statistics of such quantities is necessary in harbour design where moored ships, set-down and paddle compensation for set-down in physical models, are all of interest. For the purposes of this report, the latter two variables will be considered in detail and the application to moored ships is reserved for subsequent study.

With random seas, in both the direct study of set-down effects and in the design of wave generators, interest will focus on the probability of the extreme values which can occur. The set-down arises through non-linear interaction between the primary waves and this interaction is of the simplest type - at second order. As already mentioned, these second order interactions occur in other phenomena and are also of interest as a first approximation to more general non-linear effects. For these reasons a method will be developed here for calculating the extremes of a general process which depends quadratically on a primary gaussian process.

It will be shown that this can be turned into a special case of a theory of extremes given in a previous report (SR 3 Ref 4). This theory gave a method of calculating the mean rate of outcrossings of a general multi-dimensional gaussian process from a region of its space. Each outcrossing represented the occurrence of an extreme value and, with a further assumption of the independence of these occurrences, their probability can be derived from their mean rate.

2 THE GENERAL THEORY

The general theory, summarized here for reference purposes in the applications, concerns the mean frequency (f_s) with which a general n-dimensional gaussian process $\underline{x}(t)$ escapes from a region bounded by an (n-1) dimensional 'surface' (S). The displacements are assumed to be already reduced by scaling and rotation of the co-ordinates to a unit isotopic distribution i.e. $x_i \; x_j = \delta_{ij}$ where these brackets denote a mean value. Then taking $\dot{x}_i \; \dot{x}_j = \lambda_{ij}$ and $\dot{x}_i \; x_j = \tau_{ij} = -\tau_{ij}$ the required frequency becomes the integral over S

$$f_s = \left(\frac{1}{2\pi}\right)^{\frac{n+1}{2}} {}_s\!\int \exp - \tfrac{1}{2} \left[x_i x_j + \frac{(\tau_{ij} \; n_i \; x_j)^2}{\sigma^2 \; (\underline{n})}\right] \left\{\sigma(\underline{n}) + \tau_{ij} n_i \frac{\partial}{\partial x_j} \left[\frac{\tau_{kl} n_k x_l}{\sigma(\underline{n})}\right]\right\} ds \qquad (1)$$

where $\underline{n}$ is the unit normal to S

and $\sigma^2(\underline{n}) = (\lambda_{ij} - \tau_{ik}\tau_{jk})\, n_i n_j$

For a linear boundary $x_i n_i = r$ This integral becomes

$$f_s = \frac{1}{2\pi} \quad (\lambda_{ij}\; n_i n_j)^{\frac{1}{2}} \; e^{-\frac{1}{2}r^2} \tag{2}$$

For a spherical boundary $x_i x_i = r^2$ it reduces to the integral over the unit sphere s_1

$$f_s = \left(\frac{1}{2\pi}\right)^{\frac{n+1}{2}} r^{n-1} \quad e^{-\frac{1}{2}r^2} \int_{s_1} \sigma(\underline{n})\; ds_1 \tag{3}$$

With a more general boundary to save calculating the integral numerically it can be replaced with an asymptotic approximation by noting that the main contribution comes from the neighbourhood of the nearest point to the origin. Assuming that the point occurs on the x_1 axis and that the other coordinates are rotated to lie in the directions of the principal curvature at this point, the form of S near to the minimum is $x_1 = r - \frac{1}{2} \sum_{i=2}^{n} \kappa_i x_j^{\,2}$ The asymptotic approximation then gives

$$f_s \sim \frac{1}{2\pi} \left[\frac{\lambda_{11} - \sum_{i=2}^{n} \kappa_i r\, \tau^2_{1i}}{(1-\kappa_2 r)\;(1-\kappa_3 r)\ldots(1-\kappa_n r)}\right]^{\frac{1}{2}} \; e^{-\frac{1}{2}r^2} \tag{4}$$

3 THE ANALYTIC ENVELOPE

A narrow band random time function can be regarded as a constant carrier wave at the central frequency modified by a slowly varying envelope. This useful idea of an envelope can be extended to the general process not necessarily narrow banded by using the concept of the analytic envelope. This will not in general touch all the maximum of the process as in the narrow bounded case since for one thing subsidiary maxima develop as the process broadens, but it will nevertheless have the main property and appearance of the envelope with the fast carrier frequencies removed.

For a random signal $x(t) = \sum b_n \cos(\omega_n t + \varepsilon_n)$ (5)

let $y(t) = \sum b_n \sin(\omega_n t + \varepsilon_n)$ (6)

Then the analytic envelope is given by

$$r^2(t) = x^2(t) + y^2(t) \qquad (7)$$

Hence

$$r^2 = \sum_n b_n^2 + 2 \sum_n \sum_{m<n} b_n b_m \cos(\bar{\omega}t + \varepsilon') \text{ where } \bar{\omega} = \omega_n - \omega_m \qquad (8)$$

and r^2 contains only the difference frequencies of the spectrum.

Now

$$x^2 = \tfrac{1}{2}\sum_n b_n^2 + \sum_n \sum_{m<n} b_n b_m \left[\cos(\bar{\omega}t+\varepsilon') + \cos(\omega^+ t+\varepsilon'')\right] + \tfrac{1}{2} \sum b_n^2 \cos(2\omega_n t+2\varepsilon_n)$$

where $\omega^+ = \omega_n + \omega_m$ and for spectra which are not wide enough for the sum and difference frequencies to overlap, r^2 can be regarded as x^2 put through an ideal low pass filter. In the general case it will be seen from (5) to (6) that $y(t)$ is obtained from $x(t)$ by passing through a filter with the frequency characteristic

$$H(\omega) = -i \operatorname{sgn}(\omega)$$

The time domain equivalent of this is

$$Y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

With the impulse response $h(t) = \frac{1}{\pi t}$ (Hilbert transform) Thus, the analytic envelope is a particular ease of a quadratic function of the original signal. To use the general theory we reduce it to the standard form. Let $x(t)$ have spectrum $S(\omega)$ with moments m_n

Then
$$\overline{x^2} = \overline{y^2} = m_o \qquad \overline{xy} = 0$$
$$\overline{\dot{x}^2} = \overline{\dot{y}^2} = m_2 \qquad \overline{\dot{x}\dot{y}} = 0$$
$$\overline{\dot{x}y} = \overline{y\dot{x}} = m_1$$

Scaling the displacements by $x_1 = x/m_o^{\frac{1}{2}}$, $x_2 = y/m_o^{\frac{1}{2}}$
we have $\overline{x_i x_j} = \delta_{ij}$

$$\lambda_{11} = \lambda_{22} = \frac{m_2}{m_o}\ ;\ \lambda_{12} = 0;\ \tau_{12} = \frac{m_1}{m_o}$$

Upcrossings of the envelope value r are given by escapes from the circle $x_1^2 + x_2^2 = r^2/m_o$

Hence using (3)

$$f_s = \frac{1}{(2\pi)^{3/2}} \frac{r}{m_o^{\frac{1}{2}}} e^{-r^2/2m_o} \int_{s_1} \sigma(\underline{n})\, ds_1$$

where s_1 is the unit circle.

But $\sigma^2(\underline{n}) = (\lambda_{ij} - \tau_{ik})\, n_i n_j = \left(\frac{m_2}{m_o} - \frac{m_1^2}{m_o^2}\right)(n_1^2 + n_2^2)$
and $n_1^2 + n_2^2 = 1$

$$\text{Hence} \quad f_s = \frac{1}{(2\pi)^{\frac{1}{2}}} \left(\frac{m_2}{m_o} - \frac{m_1^2}{m_o^2}\right)^{\frac{1}{2}} \frac{r}{m_o^{\frac{1}{2}}} e^{-r^2/2m_o} \qquad (9)$$

Comparing this with the normal upcrossing frequency for values of x

$$f_x = \frac{1}{(2\pi)^{\frac{1}{2}}} \left(\frac{m_2}{m_o}\right)^{\frac{1}{2}} e^{-x^2/2\, m_o}$$

it will be seen that the gaussian distribution for the signal has been replaced as would be expected by the Raleigh distribution for the envelope but also that the zero up crossing frequency has been replaced by a lower frequency which is like a standard deviation measuring the spread of the spectrum.

4 SET-DOWN BENEATH THE WAVE GROUPS

In linear wave theory, the sea is a superposition of wave trains over a range of frequencies. In taking account of the non-linearity of the waves at the next approximation to second order these waves interact to produce 'set-down' a low frequency disturbance at difference frequencies of the primary wave train. Although relatively small this set-down can become important in some problems by exciting low frequency resonances.

For a primary wave train in shallow water of depth d,

$$\eta^{(1)} = \sum a_n \cos(\omega_n + \varepsilon_n)$$

The set-down is given by

$$\eta^{(2)} = -\frac{3g}{2d^2} \sum_n \sum_{m<n} \frac{a_n}{\omega_n} \frac{a_m}{\omega_m} \cos(\bar{\omega}t + \varepsilon')$$

Comparing this with equation (8) it will be seen that that set-down is related to the analytic envelope. The main difference lies in the ω terms in the denominater indicating that the first order signal is subject to an initial integrature. Hence defining

$$Y(t) = \int \eta^{(1)}(t)\, dt = \sum b_n \sin(\omega_n t + t_n)$$

$$\text{where } b_n = \frac{a_n}{\omega_n}$$

Equation (8) shows that $r^2 = 2m_o - \frac{4d^2}{3g} \eta^{(2)}$
Where m_n denotes the moments of the spectrum of y(t)
Hence the upcrossings of $\eta^{(2)}$ are related to those of r. If the spectral moments of $\eta^{(1)}$ are denoted by M_n then the integration means these are related to those of Y by $m_n = M_{n-2}$

Therefore from equation (8)

$$F_s(-\eta^{(2)}) = \frac{1}{(2\pi)^{\frac{1}{2}}} \left(\frac{M_o}{M_{-2}} - \frac{M_{-1}^2}{M_{-2}^2}\right) \left(2 + \frac{4d^2 \eta^{(2)}}{3g\ M_{-2}}\right)^{\frac{1}{2}} e^{-\left(1 + \frac{2d^2\ \eta^{(2)}}{3g\ M_{-2}}\right)}$$

The formula for the mean frequency has in these two cases come out explicitly. This is because they are the sums of squares of two variables one of which is obtained from the other by a filter making the two variables uncorrelated and of equal power. As the set-down case shows a filtering of the input variable makes no difference. This can be generalized to the case of any number of variables obtained by uncorrelating and equal power filters, the result now being obtained from the spherical boundary formulae (3) in many dimensions. However once this simplicity is lost, in particular by filtering after the quadratic operation, then there does not seem to be any intermediate case which is simpler to treat then the general quadratic process. Therefore this general case will be dealt with next.

5 THE GENERAL QUADRATIC PROCESS

In the theory of second order effects applied to the sea state or ship motion it has been usual to think of the corresponding quadratic process in frequency terms. Here, the amplitudes of pairs of frequencies of the primary process are multiplied together and scaled by a frequency dependent factor to provide their contribution to the second order process at the sum and difference frequencies. For the purposes of this theory however it is necessary to think of the transformation in the time domain.

The general linear filter $x(t) \rightarrow y(t)$ can be represented in the time domain by

$$y(t) = \int_0^\infty h(\tau)\ x(t - \tau)d\tau$$

where $h(\tau)$ is the impulse response of the filter. Similarly the general quadratic filter $Z(t) \rightarrow w(t)$ is represented by

$$w(t) = \int_0^\infty \int_0^\infty h(\tau_1, \tau_2)\ Z(t-\tau_1)\ Z(t - \tau_2)\ d\tau_1\, d\tau_2 \quad (10)$$

where $h(\tau_1, \tau_2)$ is an analagous double impulse response. Thus in real time the output of a linear filter is a weighted sum of past values of the input and the output of a quadratic filter is a weighted sum of products of past values.

The integral in (10) is reduced to a form needed to apply the general theory by replacing it with a finite sum

$$w(t) = h_{ij}\ Z_i(t)\ Z_j(t) \quad (11)$$

where $Z_i(t) = Z\ (t-(i-\frac{1}{2})\tau) \quad i,j = 1,2\ \ldots\ n$

where τ is a small fixed time intervaland h_{ij} is the double integral of h over the corresponding intervals.

Thus the problem is to find the mean frequency of escapes of the n dimensional gaussian random process Z(t) from the quadratic surface given by (11) with w = constant.

To apply the general theory, the Z_i must be replaced by uncorrelated variables. For any pair of variables ξ_i, ξ_j, their correlation can be expressed in terms of their cross spectrum by

$$\xi_i \xi_j = R_e \int_o^\infty S_{ij}(\omega) d\omega$$

Let $S(\omega)$ be the spectrum of Z(t)

Since $Z_j(\omega) = e^{i\omega(j-k)\tau} Z_k(\omega)$

and $\dot{Z}_j(\omega) = i\omega Z_j(\omega)$

$$Z_i Z_j = r_{i-j} \quad \dot{Z}_i \dot{Z}_j = P_{i-j} \quad Z_i \dot{Z}_j = q_{i-j}$$

where $r_k = r(k\tau)$ $P_k = p(k\tau)$ $q_k = q(k\tau)$

and $r(t) = \int_o^\infty \cos \omega t \, S(\omega) d\omega$ $P(t) = \int_o^\infty \omega^2 \cos \omega t \, S(\omega) d\omega$
$q(t) = \int_o^\infty \omega \sin \omega t \, S(\omega) d\omega$
r(t) is the autocorrelation function of Z(t)
p(t) is the autocorrelation function of $\dot{Z}(t)$
q(t) is the cross correlation function of Z(t), $\dot{Z}(t)$

Now introduce the unit isotropically distributed variables x_i $x_i x_j = \delta_{ij}$ by the equations

$Z_i = a_{ij} x_j$ where a_{ij} is a lower diagonal matrix
i.e $a_{ij} = 0$ for j i

Then $r_{i-j} = Z_i Z_j = a_{ik} a_{j\ell} x_k x_\ell = a_{ik} a_{jk}$
Since the matrix is lower diagonal these equations can be successively solved for a_{11}, a_{21}, a_{31}, a_{32}, a_{33} etc. For instance the first few equations are

$$a^2_{11} = r_o$$

$$a_{21} = r1 \quad a^2_{21} + a^2_{22} = r_o$$

$$a_{31}a_{11} = r2 \quad a_{32}a_{22} + a_{31}a_{21} = r1 \quad a^2_{31} + a^2_{32} + a^2_{33} = r_o$$

Let the inverse matrix of a_{ij} be b_{ij} so that $a_{ik} b_{kj} = \delta_{ij}$ then b_{ij} is also lower diagonal and can be solved successively.

Now $x_i = b_{ij} Z_j$

$$\text{Hence } \lambda_{ij} = \dot{x}_i \dot{x}_j = b_{ik} b_{j\ell} \dot{Z}_k \dot{Z}_\ell = b_{ik} b_{j\ell} P_{k-\ell} \quad (12)$$

$$\text{and } \tau'_{ij} = x_i \dot{x}_j = b_{ik} b_{j\ell} Z_k \dot{Z}_\ell = b_{ik} b_{jk} q_{k-\ell} \quad (13)$$

Under the transformation the surface (11) becomes

$$w = g_{ij} x_i x_j \quad (14)$$

where $g_{ij} = h_{k\ell} a_{ki} a_{\ell j}$

To calculate the asymptotic approximation the nearest point to the origin needs to be found, and this is done by finding the eigen values and unit eigen vectors of g. Since this matrix is symmetric the eigen values are real and the eigen vectors orthogonal. Suppose they are μ_i with corresponding vectors C_{ij}. Then taking the eigen vectors as a new basis with coordinates x'_i the transformation of coordinates is given by

$$x'_i = C_{ij}\ x_j \tag{15}$$

The surface S now becomes

$$w = \mu_i\ x^2_i \tag{16}$$

The nearest approach to the origin will be given by the largest eigen value. Suppose this is μ_1 then the nearest point is $x_1 = r \quad x_i = 0 \quad i\ 1$ where $r^2 = \frac{w}{\mu_1}$. In the neighbourhood of this point the surface has the form

$$x_i = \left[r^2 - \sum_{i=2}^{n} \frac{\mu i}{\mu_1} x_i{}^2\right]^{\frac{1}{2}}$$

$$\simeq r - \tfrac{1}{2} \sum_{i=2}^{n} \frac{\mu i}{\mu_1 r}\ x_i{}^2$$

Hence the principal curvatures are given by $\kappa_i r = \frac{\mu i}{\mu_1}$.

The velocity correlations transform in the new coordinates using (15) into

$$\lambda'_{ij} = C_{ik}\ C_{jk}\ \lambda_{k\ell}$$

$$\tau'_{ij} = C_{ik}\ C_{j\ell}\ \tau_{k\ell}$$

So that finally the outcrossing frequency as given in the asymptotic approximation by (4) is

$$f_s \sim \frac{1}{\pi}\left[\frac{\lambda'_{11} - \sum_{i=2}^{n} \frac{\mu_i}{\mu_1} \tau'^2_{1i}}{(1-\frac{\mu_2}{\mu_1})(1-\frac{\mu_3}{\mu_1})\ldots(1-\frac{\mu_n}{\mu_1})}\right]^{\frac{1}{2}} e^{-\frac{1}{2}\frac{w}{\mu_1}} \tag{17}$$

Where the result has been doubled to include the distribution from the symmetric near point on the opposite side of the origin.

If there are negative eigen values then negative values of w can occur and these should be treated in similar fashion but taking the largest negative eigen value to give the near point. The simplest case is in two dimensions where if both eigen values are positive the boundary surface is an ellipse but if one is negative it becomes a hyperbolm with negative values of w occurring on the opposite branch to the positive ones. The extremes of w therefore occur with quite different probabilities for the two signs.

It will be seen that the asymptotic approximation (12) breaks down if more than one eigen value equals the greatest value. This is because in such a case the sphere of minimal radius has more than a single point contact with the surface (16). In the report SR 3 (Ref 4) this possibility was noted but the various cases were thought to be too numerous for a general treatment. Here, however, with a general quadratic form for the surface the treatment can be carried further.

Suppose then that the greatest m eigen values are all equal or so nearly equal that the previous asymptotic approximation becomes of dubious value. $\mu_1 = \mu_2 = \ldots = \mu_m$. Then the sphere of minimum radius touching the surface actually touches it over the sphere of dimension (m-1)

$$x_1^2 + x_2^2 + \ldots + x_m^2 = r^2$$

The simplest case to think of is a prolate spheroid in 3 dimensions which is touched by the sphere all the way round a circle. The asymptotic approximation is then carried out only over the dimensions at right angles to these for a general point on the sphere and then the result is integrated over the sphere.

The result of doing this is to give

$$f_s \sim \frac{1}{(2\pi)^{\frac{m+1}{2}}} \frac{\left(\frac{w}{\mu_1}\right)^{\frac{m-1}{2}} e^{-\frac{1}{2}\frac{w}{\mu_1}}}{\left[(1-\frac{\mu_{m+1}}{\mu_1})\ldots(1-\frac{\mu_n}{\mu_1})\right]^{\frac{1}{2}}} \int_{s_1} \left[\sigma^2(\underline{n}) + \sum_{i=m+1}^{n} (1-\frac{\mu_i}{\mu_1})(\tau_{ji} n_j)^2\right]^{\frac{1}{2}} ds_1$$

where s_1 is the unit sphere of dimension (m-1)

This should be compared to equation (3). In both cases a quadratic function is integrated over the sphere and it is shown in SR 3 that for a circle it reduces to an elleptic integral. The integral is independent of w and so needs evaluating only once.

6 SET-DOWN COMPENSATION

Linear random waves generated by a wave paddle inevitably carry the second order set-down water motion with them together with further second order terms generated by the finite motion of the paddle. If the paddle motion corresponding to these water motions is not accounted for in the driving signal to the paddle, then the paddle will generate spurious low frequency free waves whilst negating them at the paddle face (Ref 3). To stop this happening it is necessary to compute a second order signal from the primary paddle signal and add it into the paddle motion as set-down compensation.

For a piston paddle in shallow water this turns out to be relatively easy to do since the second order signal w(t) can be obtained to a good approximation from the first order one Z(t) simply by squaring and integrating.

$$w(t) = -\alpha \int Z^2(t)\, dt$$

In practice to avoid difficulty with the D.C. the integration operation is replaced by a filter which approximates the frequency characteristic of integration only in the range above a certain minimum frequency where the set-down effects are judged to be important. Below the minimum frequency the filter characteristic drops away to zero at D.C. Transferring the frequency characteristic into the time domain impulse response the second order signal computation becomes

$$w(t) = \int_0^\infty h(\tau)\, Z^2(t-\tau)\, d\tau$$

Again replacing this by a finite sum it becomes a special case of equation (11)

$$w(t) = h_i\, Z^2{}_i(t) \qquad (18)$$

The theory developed above can therefore be directly applied to find the extremes of the second order signal. Although this case seems simpler in that (18) is already in a diagonal form the gain is marginal for once the transformation to uncorrelated variables is made as in (14) the diagonal form is lost.

7 CONCLUSIONS

1. The prediction of extremes of a general quadratic process has been shown to be a special case of a theory developed in a previous report (Ref 4). As a result, the mean frequency of upcrossings of any given extreme level, can be defined. And, with the assumption of the independence of these

extreme events, their probability of occurrence can be derived.

2. The technique has been applied to two cases of interest. One concerns the prediction of extreme values of set-down beneath wave groups in random seas and the other concerns the signal to shallow water wave-makers where compensation for set-down is required to avoid the generation of spurious long waves. In the latter case, the extremes of paddle movement with set-down compensation have to be capable of prediction to enable the wave generator to be properly designed (Ref 3).

8 REFERENCES

1. Bowers E.C. Long period oscillations of moored ships subject to short wave seas.

2. Bowers E.C. Harbour resonance due to set-down beneath wave groups. J Fluid Mech Vol 79 p.71, 1977

3. The correct representation of set-down beneath wave groups in random wave harbour modelling HR Report SR 101, 1986

4. Gilbert G Extremes of a general Gaussian process HR Report SR 3, 1985