Sediment Transport in Channels: an alternative approach

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SEDIMENT TRANSPORT IN CHANNELS :
AN ALTERNATIVE APPROACH

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SYNOPSIS

The relationship of sediment transport to fluid flow is considered. Physical reasoning leads to dimensionless groupings of the variables which are different for coarse sediment and for fine sediment, because of dissimilar modes of transport. This concept provides a basis for a new analysis of data from flume experiments, and a method for dealing with transitional sizes of sediment is suggested. The analysis supports the theory put forward, and tentative predictive equations are derived which relate total bed sediment flux to measurable properties of flow.

This is the first report on a continuing investigation and it is recommended that the next report on an extended analysis is awaited before applying the results.

INTRODUCTION

Many theories have been put forward in attempts to provide frameworks for the analysis of data on sediment transport, some being based on the physics of particle motion and others on similarity principles or dimensional arguments. Many equations have been proposed as a result of these analyses, of varying degrees of complexity, yet even the more sophisticated of the procedures and formulae remain open to question. It is well known that very different answers may result from the use of these alternative methods, and there is at present no really sound basis on which any one could be selected to the exclusion of the others.

The transport of non-cohesive sediments by a steady uniform flow of fluid in an open channel is a complex process, and the physics of this two-phase motion is as yet incompletely understood. The engineering aspect of the subject, namely the prediction of the sediment transport rate from a knowledge of the flow parameters (or vice versa), is not necessarily advanced by increasing complexity in the computational procedures to be followed.

There has been an academic preference for shear stress as the main parameter defining the stream's transporting power. However, the total shear on a deformed bed (rippled or duned) is in part composed of the along-stream components of the normal pressures on the irregular bed profile. As these normal pressures will not contribute directly to sediment motion, most current methods separate the bed shear into the non-transporting form loss and the shear on the grains. As the rate of transport is very sensitive to transporting power, any inaccuracy in this separation procedure would give large errors of prediction. In engineering practice, this factor is of great importance because few natural streams or irrigation channels in alluvial sand have a plane bed.

Several researchers have suggested that shear stress is not the most convenient, nor the most rational, basis of a sediment transport function, and have proposed methods of correlation that use average stream velocity in preference to shear stress.

The purpose of the present paper is to develop and examine a new framework for the analysis of transport data. If successful, this will avoid refinements which complicate the application without adding much to accuracy; the advantages of dimensional analysis will be incorporated, but physical arguments will be used in deriving the form of the functions to be tested; the variables will be directly related to those the engineer can readily visualise and measure; and the uncertainty of slope separation procedures will be avoided.

So many reviews of previous work have been published, either in their own right or as introductions to new work, that no apology is necessary for not including such a review in this paper. An unfortunate result of such an omission is the loss of the opportunity for acknowledging by name many previous workers in sediment research. The writer apologises for his failure to make individual acknowledgement of the ideas in the literature which have been developed to form the present treatment of the subject.

General Approach

The procedures to be described side-step slope separation and allowance for wall shear at the outset, being based on mean stream velocity and gross energy degradation. The former seems appropriate for coarse sediments, and the latter for fine sediments, with a transition between these zones where the desirability of including both measures of transporting capacity will be examined.

Physical considerations lead to certain preferred groupings of the independent variables, with one of the non-dimensional parameters being a notional efficiency of transport, i.e. the ratio of the rate at which work is done to the power expended. The dimensionless groups that finally emerge have the advantage of separating the engineering variables, as well as the main quantities measured in typical flume experiments. Thus they have the merit of convenience as well as avoiding spurious correlations.

After deducing the framework of analysis, certain coefficients are evaluated by reference to experiments on coarse sediments, and a particular set of experimental data for a single sediment size is used to suggest a form of equation for further examination. This then permits the analysis of intermediate sediment sizes to proceed on a firm statistical basis, the empirical coefficients relevant to the transition range of materials being deduced on the basis of minimising scatter. The extent to which the initial hypotheses are supported, in the sense of providing a sound basis of data analysis, can then be reviewed, and limitations established.

THEORETICAL ANALYSIS

Coarse sediment

The definition of a coarse sediment will be deferred for the present except for the premise that the transport of such material is a "bed" process; in other words when coarse sediments move they remain close to the bed. The agency under which they move is the velocity field existing at the granular surface, which determines the shear stress on the grains. If ripples or dunes exist, the time-average shear stress will vary from point to point, being a maximum at the crests and a minimum in the troughs of the bed features. Therefore average values, both spacial and temporal, are considered, distinguishing between the true shear stress and the components in the direction of motion of normal pressures on the irregular bed. Thus

$$\tau_{cg} \neq \rho gmi \qquad .....(1)$$

*Standard terminology is used as far as possible; definitions are given at the end of the paper; (the suffix cg denotes coarse grain conditions). The effective shear, so far as the transport of coarse material near the bed is concerned, is assumed to bear a similar relationship to mean stream velocity as in the case of a plane granular surface at rest.

For rough-turbulent flow:

$$\frac{V}{v_{cg}} = \sqrt{32} \log \left(\frac{12.3y}{k_s}\right) \qquad .....(2)$$

where v_{cg} is the shear velocity ascribed to the grains, defined as $\sqrt{(\tau_{cg}/\rho)}$, and k_s is a linear measure of the grain roughness.

More generally,

$$\sqrt{\left(\frac{\tau_{cg}}{\rho}\right)} = \frac{V}{\sqrt{32}\log(\alpha y/D)} \qquad \ldots\ldots(3)$$

where α is a numerical constant, incorporating both the factor 12.3 that appears in eqn 2 and a constant of proportionality relating k_s to the median sediment diameter D.

The shear stress given by eqn 3 acts on a single layer of grains, their resistance to sliding or rolling motion being a result of their immersed weight and a coefficient of friction, tan ϕ. The resistance of a unit area of grains is thus

$$p \tan\phi . \rho g(s-1)D$$

where p is a void or packing factor. Omitting the factor p tan ϕ, which can be taken as constant for typically shaped sediments, the ratio of the applied stress τ_{cg} to this resistant stress describes the sediment mobility, which will be denoted F_{cg}^2.

Thus

$$F_{cg} = \frac{V}{\sqrt{32\, g(s-1)D}} \; \frac{1}{\log(\alpha y/D)} \qquad \ldots\ldots(4)$$

The conventional definition of a coarse grained surface is one where the flow at the bed is fully turbulent and the shear stress thereon is therefore independent of viscosity. The value of F_{cg} at which motion begins is not a function of the Reynolds number of the grain in this zone. Significant transport will exist at values of F_{cg} above this critical value, and its efficiency will depend on the value of F_{cg}:

clearly efficiency is zero at and below the critical value.

The useful work rate, per unit of plan area, is

$$g_b \gamma (\frac{s-1}{s}) \tan \phi$$

where g_b is the sediment load (weight in air per unit width per unit time) with an assumed angle of dynamic friction ϕ. The derivation of the formula above is obvious for sliding motion in contact with a plane horizontal bed, but it can also be shown to apply to an irregular bed.

The part of the stream power that generates the grain shear stress is given by $\tau_{cg}V$, per unit area of bed. Calling the efficiency of the process E_{cg}, utilising eqn 3 and omitting $\tan \phi$ on the basis that any variation in it will also be a function of F_{cg},

$$E_{cg} = g_b \gamma (\frac{s-1}{s}) \frac{32 [\log (\alpha y/D)]^2}{\rho V^3} \qquad \text{.....(5)}$$

$$g_b = X_b \rho g V y \qquad \text{.....(6)}$$

where X_b is the transport of bed sediment expressed as a concentration by weight, i.e. the mass flux of sediment divided by the mass flow rate.

Thus

$$E_{cg} = 32 \frac{gy}{V^2} \gamma (\frac{s-1}{s}) [\log (\alpha y/D)]^2 X_b \qquad \text{.....(7)}$$

For coarse sediments, it was suggested that

$$E_{cg} = f \{F_{cg}\} \qquad \text{.....(8)}$$

but a more convenient equivalent expression is

$$G_{cg} = f\,(F_{cg}) \qquad \ldots\ldots(9)$$

where

$$G_{cg} = E_{cg}F_{cg}^{\;2} = \frac{X_b y}{sD} \qquad \ldots\ldots(10)$$

G_{cg} is directly proportional to the sediment in transport per unit area of the bed, and this is a function of a mobility number depending directly on stream velocity. As this incorporates an "immersed" gravity effect and sediment diameter, it has the form of a Froude number.

Particle Reynolds number

The Reynolds number of an individual grain determines the extent to which viscous forces influence its motion. A particle Reynolds number above a certain limit denotes that viscous effects are negligible, and the grain is a coarse one; a Reynolds number below a different limit denotes a predominance of viscous effects, which is the usual definition of a fine particle in hydraulics; between these limiting values the performance of the grain is transitional. Particle Reynolds number is important in extended the analysis to fine and transitional materials.

The usual definition of the Reynolds number of a grain on the bed uses the grain shear stress (in the form of shear velocity) as the characteristic velocity, and thus may be written:

$$Re_b = \frac{VD}{\sqrt{32}\log(\alpha y/D)\,\nu} \qquad .....(11)$$

Thus the transport of grains at the bed can be described by a more general form of eqn 9, which could apply to transitional sediment sizes

$$G_{gr} = f\{F_{gr};\ Re_b\} \qquad .....(12)$$

This would not be a convenient framework of analysis because F_{gr} and Re_b both depend on $V/\log(\alpha y/D)$, so neither group remains constant during a typical set of flume experiments with a single sediment. This is readily avoided by replacing Re_b by $Re_b^{\frac{2}{3}}F_{gr}^{-\frac{2}{3}}$ which will be denoted D_{gr}.

$$D_{gr} = D\left\{\frac{g(s-1)}{\nu^2}\right\}^{\frac{1}{3}} \qquad .....(13)$$

This form of grain Reynolds number remains constant during a typical experimental series, conducted at a steady temperature. The preferred function is therefore:

$$G_{cg} = f\{F_{gr};\ D_{gr}\} \qquad .....(14)$$

The finer sediments will travel at least partly in suspension. However, the Reynolds number of a particle in suspension is not associated with shear velocity at the bed, but with its own fall velocity, i.e.

$$Re_s = \frac{wD}{\nu} \qquad .....(15)$$

The drag coefficient of a falling particle is a function of Re_s. At the terminal velocity, drag equals the immersed weight, and hence (omitting numerical coefficients)

$$\frac{\rho g (s-1) D^3}{\rho w^2 D^2} = f\left(\frac{wD}{\nu}\right)$$

These dimensionless groups may be combined to yield

$$\frac{g (s-1) D^2}{w\nu} = f\left\{\frac{g (s-1) D^3}{\nu^2}\right\} \qquad .....(16)$$

i.e. $$w = \frac{g (s-1) D^2}{\nu} f(D_{gr}) \qquad .....(17)$$

This may be compared with Stokes law for fine particles,

$$w = \frac{g (s-1) D^2}{\nu} \qquad .....(18)$$

It is thus seen that the D_{gr} - form of particle Reynolds number is valid for describing the departure of fall velocity from Stokes law in the transition range of particle sizes, and hence it is relevant to <u>all</u> phases of sediment transport.

Fine sediment

Although it was suggested that the transport of coarse sediment was primarily a bed process, fine sediments travel largely in suspension, and the turbulent intensity which provides the motive power is a function of the total shear on the bed. Assuming the ratio of channel width to depth to be large enough for the bulk of the shear to be carried by the bed:

$$\tau_{fg} = \rho g y i \qquad .....(19)$$

The stream power, per unit plan area, is now given by $\tau_{fg} V$.

The work done in keeping the sediment in suspension is:

$$X_s \frac{(s-1)}{s} \rho g y w$$

where w is the fall velocity

Thus

$$\text{efficiency} = X_s \frac{(s-1)}{s} \frac{y\, g\, w}{V\, v_{fg}^2} \qquad .....(20)$$

where v_{fg} is the shear velocity defined as $\surd(\tau_{fg}/\rho) = \surd(gyi)$

For small particles, Stokes' law applies, and

$$w = \frac{g\, D^2\, (s-1)}{18\nu}$$

Omitting the numerical constant,

$$E_{fg} = X_s \frac{(s-1)^2}{s} \frac{y\, g^2\, D^2}{\nu\, V\, v_{fg}^2} \qquad .....(21)$$

The treatment of particle mobility for coarse material is now inappropriate. Instead it is assumed to be given by the ratio of shear velocity, v_{fg}, to fall velocity, w.

Thus

$$F_{fg} = \frac{v_{fg} \ \upsilon}{gD^2(s-1)} \qquad .....(22)$$

It will be remembered that a two-variable function for sediment transport emerged from the theory for coarse grains (eqn 8). Similarly, for fine material we would expect

$$E_{fg} = f\ (F_{fg}) \qquad .....(23)$$

In order to eliminate both g(s-1) and ν from the dimensionless sediment transport parameter, and bearing in mind the need to cover the transition zone, it is convenient to generalise the above to

$$E_{fg} = f\ (F_{fg},\ D_{gr}) \qquad .....(24)$$

and define

$$G_{fg} = E_{fg} F_{fg}^{3} D_{gr}^{3} \qquad .....(25)$$

This yields

$$G_{fg} = \left(\frac{X_s\ y}{s\ D}\right) \frac{v_{fg}}{V} \qquad .....(26)$$

Eqn 24 is replaced by

$$G_{fg} = f\ (F_{fg} D_{gr}^{3/2},\ D_{gr}) \qquad .....(27)$$

to form a function analogous to eqn 14 for coarse sediments.

Comprehensive function

Recapitulating:

Coarse grain zone:

$$G_{cg} = f\ (F_{cg})$$

i.e. $$\frac{X_b\ y}{s\ D} = f\left\{\frac{V}{\sqrt{g(s-1)D}}\quad\frac{1}{\sqrt{32}\ \log(\alpha y/D)}\right\} \qquad .....(28)$$

Fine grain zone:

$$G_{fg} = f\ (F_{fg} D_{gr}^{3/2},\ D_{gr})$$

i.e. $$\left(\frac{X_s\ y}{s\ D}\right)\frac{v_{fg}}{V} = f\left\{\frac{v_{fg}}{\sqrt{g(s-1)D}}\ ;\ D_{gr}\right\} \qquad .....(29)$$

Noting the similarities between eqn 28 and 29, we may write a function which comprises both, as follows:-

$$G_{gr} = f\ (F_{gr}\ ;\ D_{gr}) \qquad .....(30)$$

where

$$G_{gr} = \left(\frac{X\ y}{s\ D}\right)\left(\frac{v_*}{V}\right)^n \qquad .....(31)$$

$$F_{gr} = \left(\frac{V}{\sqrt{g(s-1)\ D}}\quad\frac{1}{\sqrt{32}\ \log(\alpha y/D)}\right)\left(\frac{v_*\ \sqrt{32}\ \log(\alpha y/D)}{V}\right)^n \qquad .....(32)$$

$$D_{gr} = \left(\frac{g(s-1)}{\nu^2}\right)^{\frac{1}{3}} D \qquad .....(33)$$

Where n = 0 for coarse material, and n = 1 for fine material.

As the need to distinguish between fine and coarse material has ended with the derivation of a common function, the conventional nomenclature for overall shear velocity in a wide channel, $v_* = \sqrt{g\,y\,i}$, replaces v_{fg}.
It is to be expected that the above functional form will apply through the transition range of sediment sizes, with $0<n<1$, with X referring to the total transport of bed material, irrespective of whether it is bed load (usually regarded as sediment in contact with and saltating just above the bed) or bed material travelling largely in suspension.

Equations 30 to 33 provide the required comprehensive framework for data analysis, the anticipated type of family of curves being as shown in Fig. 1. We have yet to determine empirically a suitable value for the coefficient α, the form of the function (i.e. the shape and disposition of curves in Fig. 1) and how n varies with D_{gr}. In fig 1a, the family of curves consist of iso-D_{gr} lines, with F_{gr} as abscissa and G_{gr} as ordinate. This is the basic form for analysing experimental data, each test series being at a constant value of D_{gr}. In fig 1b, the same functional relationship has been shown as iso-G_{gr} lines with D_{gr} as abscissa and F_{gr} as ordinate. The lowest line on this, representing a trivial rate of transport, is loosily akin to the Shields(1) initial motion curve, the upper lines being for a progressively more active bed.

ANALYSIS OF FLUME DATA

Coarse sediment

Because a much simpler relationship is expected for coarse sediment than for intermediate sizes, this zone will be examined first. As α occurs within a logarithmic term, the value of F_{gr} is not strongly dependent upon the value ascribed to α. In order to optimise its effect in allowing for the relative depth y/D, data covering a wide range of relative depths are required. Most test series with well-established motion cover a rather narrow range of depths, so Neill's data on the initial motion of gravels in the range 6.2 mm < D < 28.1 mm were utilised [2]. These cover a 200-fold range of y/D.

By eqn. (3), initial motion is defined by

$$0 = f\left\{F_{cg}\right\}$$

i.e. $$\frac{V}{\sqrt{32g(s-1)D}} = \text{constant} \times \log(\alpha y/D) \qquad \ldots\ldots(34)$$

This is plotted in Fig. 2a, on a log linear basis. The expected relationship of $V/\sqrt{g(s-1)D}$ to D/y would plot as a straight line, the intercept of which on the x-axis defines the value of α ($\log(\alpha y/D) = 0$). The constant is given by the intercept on the vertical line, $\log(\alpha y/D) = 1.0$. An overall fit to all the plotted points is shown:

$$\frac{V}{\sqrt{32g(s-1)D}} = 0.095 \log\left(\frac{1000y}{D}\right) \qquad \ldots\ldots(35)$$

This is an unrealistically high value of α but it appears that the data really form two groups depending on whether the sediment considered is above or below 15 mm. Although there is appreciable scatter, each group approximates to

$$\frac{V}{\sqrt{32g(s-1)D}} = \text{constant} \log\left(\frac{10y}{D}\right) \qquad .....(36)$$

(the lines go through the point D/y = 10, $V/\sqrt{g(s-1)D} = 0$)

The constant varies from 0.19 to 0.23, depending on whether the gravel is fine or coarse, but there is not a systematic variation with diameter. The sizes do not plot in a sequence of curves, and this suggests that the "constant" varies because of some other peculiarity. Factors such as sediment shape or packing, or subjective definition of initial motion might be relevant but it seems more probable that the change in the "constant" occurs because in this particular test series the flow was supercritical for the three coarsest sizes and sub-critical for the three finer ones. An interaction between the bed roughness and the free-surface could change the apparent mobility of a sediment. Lines denoting Fr values of 0.8, 1.0 and 1.25 have been added to Fig. 2a to illustrate this factor.

In view of the closeness of the α value in eqn. 36 to the "standard" value of 12.3 in the rough turbulent equation, it is reasonable to proceed with the analysis on the basis that $\alpha = 10$, but it is stressed that this is an approximate value chosen for convenience. The data are too scattered to provide a refined estimate.

Fig. 2b relating to glass spheres and cellulose acetate balls lends support to a value of 10, but again there is some variation of the "constant" for supercritical flow.

One of our general aims is to establish the D_{gr} value that separates coarse sediment from transitional sizes, but a preliminary estimate of the limit is needed (based on previous treatments of sediment transport, and boundary friction on immobile surfaces) in order to assemble and consider transport data. A median sand diameter of between 1 mm and 2 mm is appropriate, i.e. a D_{gr} value between 25 and 50 will probably form the limit between the transition and rough zones (for sand in water at 15°C, $D_{gr} \simeq 25 D$ with D in mm).

In order to avoid an upper phase of sediment transport, which the present theory is not intended to cover, no flume data for channel Froude numbers approaching or exceeding unity were included in the subsequent analysis: an upper limit of $Fr = 0.8$ was applied in selecting data.

The data for coarse material were from the following sources:

(a) U.S. Waterways Experiment Station, sand 9, 4.1 mm[3]

(b) Liu, sands II, III, IV and VI, diameters 3.4 mm, 2.3 mm 1.4 mm, 1.8 mm[4, 5]

(c) Williams, sand of diameter 1.35 mm[6]

Fig. 3 shows these data plotted separately, the parameters being determined from equations 31, 32 and 33, with n = 1. It is clear that all are remarkably similar.

The curves approach a limiting value of F_{gr} asymptotically at low values of the transport parameter G_{gr}. This asymptote therefore represents nominal initial motion, sufficiently well defined by a trivial transport of sediment corresponding to $G_{gr} \sim 10^{-4}$. This asymptotic value of F_{gr} is denoted A, and Fig. 3 shows that it is approximately 0.177. There is a variation of up to ten per cent either side of the mean value, as in Table 1. The figure of 0.19 that emerged from considering Neill's initial motion tests in the sub-critical range is within the same range of values.

The recent tests by Williams have many features to commend them, and it is significant that they show rather less scatter than some of the other results. They are particularly valuable therefore in considering an explicit functional equation. The type of relationship suggested by Fig. 3 is a power function of G_{gr} with $(F_{gr} - A)$. This is examined in the log-log plot of Fig. 4a, where the Williams data is shown by bold open circles. They lie close to a line with slope 1.5, i.e. for

$$10^{-4} < G_{gr} < 2 \times 10^{-2},$$

$$G_{gr} = 0.338 (F_{gr} - A)^{1.5} \qquad .....(37)$$

where A is as given in Table 1.

Although there is reasonable agreement between the general curves plotted in Fig. 3, the type of plot in 4a is very searching: F_{gr}-A = 0.02A represents a 2 per cent increase in velocity compared with the nominal start of motion, and F_{gr}-A = 0.1A is a 10 per cent increase. Thus quite small errors in the measurement of average flow depth over an irregular bed cause a big change in the plotting position.

For the coarser sediments ($D_{gr} \geqslant 34$), equation 37 expressed in engineering terms becomes:

$$X = 0.338 \frac{sD}{y} \left\{ \frac{V}{\sqrt{32g(s-1)D}} \quad \frac{1}{\log(10y/d)} \quad -0.177 \right\}^{1.5} \qquad .....(38)$$

No change in the transport relation seems necessary as a result of a change in bed form from plane to rippled or duned (higher stages of bed development have been excluded from the analysis). This is supported by reference to Fig. 3, where different symbols are used to denote bed-form. It is apparent then that a negligible rate of transport is defined by:

$$V_o = \sqrt{g(s-1)D} \quad \log\,(10y/d) \qquad .....(39)$$

This applies to sands and gravels of diameter 1.35 mm and above with sub-critical flow in the channel.

The analysis confirms that sediment transport is very sensitive to mean stream velocity, especially when velocity is not much above that which initiates motion.

Intermediate sediment sizes (transition zone)

Equations 30 to 33 provide the framework for the analysis of flume data for intermediate sediment sizes. An optimisation procedure is required to derive the value of n which minimises the scatter in the data, and the statistical method depends on an explicit function to examine. It was initially assumed that the form of the transport function would remain similar to that in the coarse zone, i.e. the curve relating G_{gr} to F_{gr} for a given value of D_{gr} will have the same shape on a log-log plot. The general function is therefore derived from eq. (37), by inserting the coarse-grain value of A, and then permitting a shift of the curve by introducing a coefficient β.

$$G_{gr} = 0.338\ (\beta F_{gr} - 0.177)^{1.5} \qquad .....(40)$$

where

$$G_{gr} = \frac{Xy}{sD}\left(\frac{v_*}{V}\right)^{n} \qquad .....(41)$$

$$F_{gr} = \frac{V}{\sqrt{g(s-1)D}\ \sqrt{32}\ \log(10y/D)}\left(\frac{v_*\sqrt{32}\ \log(10y/D)}{V}\right)^{n} \qquad .....(42)$$

$$\text{and } \beta = f(D_{gr}) = f\left\{\left(\frac{g(s-1)}{\nu^2}\right)^{\frac{1}{3}} D\right\} \qquad .....(43)$$

Considering G_{gr} as ordinate and F_{gr} as abscissa, the optimisation procedure minimised the horizontal displacement of points from the average line conforming with eqn. (40), on a least square basis.

The horizontal displacement of a point, bearing in mind the plot is a log-log one, is given by

$$e = \log \left\{ \frac{\beta\ F_{gr}}{\left(\frac{G_{gr}}{0.338}\right)^{\frac{2}{3}} + 0.177} \right\} \quad .....(44)$$

For any collection of data for a single sediment, the "best fit" value of the empirical constant β may be derived as follows. Eqn. 44 may be rewritten

$$e = \log \left\{ \frac{F_{gr}}{\frac{0.177}{\beta}\left[(40G_{gr})^{\frac{2}{3}} + 1\right]} \right\} \quad .....(45)$$

The computer programme was therefore designed to minimise:

$$\sum e^2 = \sum_1^M \left\{ \log\left(\frac{F_{gr}}{A(11.7\ G_{gr}^{\frac{2}{3}} + 1)} \right) \right\}^2 \quad .....(46)$$

where $\frac{0.177}{\beta}$ has been replaced by the variable A. The search for minimum error was conducted by varying both A and n, both of which were expected to depend on D_{gr}, but would take constant values in a given test series.

The search programme involved many computer runs. Effectively it plotted $\sum e^2$ against A and n, and found the A and n values giving the lowest value within the summation error contours. The search reduced the n step four times as it approaches the optimum, and then reduced the A step as a final refinement, as follows:

1st computer run:	$\Delta\ n_1 = 0.25$	,	$\Delta\ A_1 = 0.01$
2nd " "	$\Delta\ n_2 = 0.05$	,	$\Delta\ A_2 = 0.01$
3rd " "	$\Delta\ n_3 = 0.01$	,	$\Delta\ A_3 = 0.01$
4th " "	$\Delta\ n_4 = 0.002$	,	$\Delta\ A_4 = 0.01$
5th " "	$\Delta\ n_5 = 0.0004$	,	$\Delta\ A_5 = 0.002$

Fewer steps in A were needed because the position of the minimum R.M.S. error varies much less with A than with n when the mesh size is reduced. The computer program was written in ALGOL for use on the KDF 9 computer at Culham, and each run takes up to 4 mins for M<30 (number of test points).

The data processed in this way consisted of ten sets of flume experiments, conducted by Guy, Simons, and Richardson(7), with sands in the range 0.19 mm < D < 0.93 mm, together with Kennedy and Brooks(8) tests on a fine 0.14 mm sand, and the Williams(6) data for 1.35 mm sand* which was the finest grade included tentatively in the coarse-grain analysis.

The results of the computer analysis are shown in table 2, where the optimum n and A values are listed against D_{gr}. The analysis was successful in that accurate best-fit values of A and n were obtained which varied reasonably consistently with D_{gr}. n, the coefficient that distinguishes between the coarse-grain bed-process theory and the fine-grain turbulence-transport theory,

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* See footnote on page 31

varies as predicted from a value of about 1 at the fine end of the spectrum to 0 at the coarse end. As the evaluation is not subjective, it provides strong evidence in support of the theory that was propounded.

Also included in table 2 are the values of the standard deviation, S = R.M.S. error, and the equivalent percentage scatter in F_{gr}. Remembering its form, discrepancies in F_{gr} will arise from the measurement of depth, discharge per unit flume width, and median sediment diameter. The departures of the experimental points from the plot of the suggested formulae are therefore not unreasonable, even if the possibility of error in the measurement of the quantities that make up the transport parameter G_{gr} are discounted. The high reliability of the Williams data* is confirmed, but some of the finer sands give results that deviate appreciably from the best-fit equation of the curve tested.

In figure 5, the optimised values of n and A are plotted, and curves are indicated which would be suitable functions for application to sediment transport problems. It appears from this plot that Williams 1.35 mm sand is just within the transition zone, which extends from $D_{gr} = 7$ to $D_{gr} = 35$. The suggested equation for n is:

$7 < D_{gr} < 35$,

$$n = 2.21 - 1.43 \log D_{gr} \qquad \text{.....(47)}$$

$D_{gr} > 35$, $n = 0$; $D_{gr} < 7$, $n = 1.0$

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* See footnote on page 31

It is not possible, with the limited amount of data so far analysed, to define the variation of A with D_{gr} with assurance. The full-line drawn in the upper part of fig. 5 corresponds to:

$$A = \frac{1.05}{\sqrt{D_{gr}}} \qquad \text{.....(48)}$$

when $D_{gr} < 35$; $A = 0.177$ for $D_{gr} > 35$.
However this involves a discontinuity in A between the transition and coarse sizes of sediment. More accurate data is required for the transitional sand sizes in order to define this function better.

For comparison with the G_{gr} - F_{gr} plots for coarse sediment included in fig. 3, fig. 6 has been prepared from the data for transitional sized materials. The full lines are the best-fit equations based on the optimised values of n and A. This permits us to review the similarity hypothesis that led to the use of the same power function as applied to coarse material. There is evidence that the equation is a less satisfactory fit to the data for the finest of the materials considered. The plots for $D_{gr} < 7$ suggest that an exponent in equation (4) differing from 1.5 would be a more accurate fit, so that the present analysis can not be considered to have found the best solution for fine materials, $D_{gr} < 7$. (<0.3 mm sand).

An optimisation procedure that permits this exponent to vary as well as varying n and A is feasible, and is now under way, using additional data for sands of median size below 0.2 mm. The broken lines in the sections of fig. 6 for $D_{gr} < 7$ represent a rough interim solution only.

Fig. 7 contains comprehensive plots of the transition and fine zone data, in the form G_{gr} against $\frac{F_{gr}}{A} - 1$. This may be compared with the corresponding coarse grain plot of fig. 4, and it will be observed that, apart from the results for $D_{gr} < 7$, agreement with the form of equation tested is reasonably satisfactory.

DISCUSSION OF RESULTS

Initial motion

Because the quantity A represents the F_{gr} value at which transport begins, nominal initial motion can be represented in a fashion analogous to the Shields curve. This is shown in fig. 8, where A^2 is plotted against D_{gr}. Neills'[2] results for gravel beds are included in the plot, omitting the results with super-critical flow. The version of the Shields curve given by Brown[9] has also been added for comparison, taking A^2 as equivalent to Shields parameter $\frac{v_*^2}{g(s-1)D}$. The equivalence is immediately apparent for fine materials (eqn. 42, with n = 1), and is also correct for coarse materials under plane bed conditions, invoking the rough-turbulent equation.

There is a surprising discrepancy between the conventional Shields curve and a nominal initial motion condition represented by $F_{gr} = A$. The latter is effectively an extrapolation of the equation for established motion to zero motion. Not only is the fine end different in its general elevation (though not necessarily in its slope) the coarse end differs too in the reverse direction. No less significant is the apparently shorter length of transition, and the lack of any appreciable dip in the function.

The full line in fig. 8 is one possible function, corresponding to eqn. 48 for $D_{gr} < 35$ and $A = 0.177$ for $D_{gr} > 35$. However, there is clearly a good deal of uncertainty as to what the best relation might be, that could be resolved only by the acquisition and analysis of more, accurate data.

The values of A upon which fig. 8 depends are based on a function for established sediment transport which, especially at the intermediate and higher rates, implies the existence of ripples or other bed forms. This possibly explains why the A value does not agree with previous data (from Shields and other sources) for the threshold condition when the bed is plane. Minimal transport under plane bed conditions would not follow the equation that was deduced for $G_{gr} > 10^{-4}$.

Established motion

The form of function envisaged in fig. 1 anticipated that the curves would overlap, a feature that seemed likely bearing in mind the dip in the conventional Shields curve. This dip having apparently been discounted by the present analysis, and the function through the transition zone being approximately similar to that in the coarse zone, the family of curves that finally emerges is somewhat simpler than fig. 1.

Fig. 9 shows the derived equation plotted in the form of iso-D_{gr} lines on F_{gr} and G_{gr} axes. Fig. 10 shows iso-G_{gr} curves with F_{gr} and D_{gr} as axes. The broken lines for $D_{gr} < 7$ acknowledge the uncertainty in the fine-grain function.

Recapitulating, the equations are:

$$G_{gr} = 0.025 \left(\frac{F_{gr}}{A} - 1 \right)^{1.5} \qquad .....(49)$$

where

$$G_{gr} = \frac{Xy}{sD} \left(\frac{v_*}{V} \right)^{n} \qquad .....(50)$$

$$F_{gr} = \frac{v_*^{\,n}}{\sqrt{g(s-1)D}} \left(\frac{V}{\sqrt{32}\,\log(10y/D)} \right)^{1-n} \qquad .....(51)$$

n and A are functions of D_{gr}

$$D_{gr} = \left(\frac{g(s-1)}{\nu^2} \right)^{1/3} D \qquad .....(52)$$

$D_{gr} > 35$:

$$A = 0.177 \qquad .....(53)$$

$$n = 0 \qquad .....(54)$$

$7 < D_{gr} < 35$:

$$A = \frac{1.05}{\sqrt{D_{gr}}} \qquad .....(55)$$

$$n = 2.21 - 1.43 \log D_{gr} \qquad .....(56)$$

$D_{gr} < 7$

$$A = \frac{1.05}{\sqrt{D_{gr}}}$$

$$n = 1.0 \qquad .....(57)$$

Because of the discontinuous equation used for A, the sediment curves in fig. 10 show a discontinuity as well, but the type of adjustment that would arise if a continuous function, such as the chain-dotted line in fig. 5, was used instead is easily envisaged. It is to be expected that further analysis will produce such a continuous function, and will further amend the shape of the curves for the fine zone because of the need for an exponent in eqn. 49 which varies when $D_{gr} < 7$.

CONCLUSIONS

This treatment of the transport of bed-material is based on the concept that predictive equations for coarse sediment and for fine sediment might with advantage be based on different dimensionless groupings of the dependent and independent variables. The different groups would reflect the dissimilar modes of transport, the transport of coarse sediment being primarily a bed process linked to nett grain shear and the transport of fine sediment being largely in suspension, when gross shear velocity and the sediment fall velocity are the most significant variables.

Physical arguments were used to deduce dimensionless parameters, and the hypothesis was made that the transition zone between coarse and fine sediments could be accommodated by phasing the one preferred group of parameters out as the other group was phased in. An analysis of laboratory experiments tested the method, with promising results. The detailed conclusions that emerged are as follows:

1. Data for coarse material covering a wide range of values of y/D confirm that initial motion is described by

$$F_{cg} = \text{constant}$$

where
$$F_{cg} = \frac{V}{\sqrt{32\, g(s-1)D}} \quad \frac{1}{\log(\alpha y/D)} \qquad \ldots\ldots(58)$$

2. Provided the flow is sub-critical, the threshold value of F_{cg} is about 0.177, and α is approximately 10. (This value was then used in the rest of the analysis).

3. The description "coarse" may be applied to sands and gravels above 1.4 mm median size ($D_{gr} > 35$).

4. The flume data chosen for analysis plot as a single function, thus confirming the form of equation 9.

5. The one relationship covers both plane bed conditions and those sub-critical states of flow involving bed features.

6. For coarse sediment, transport can be expressed as a power function of the velocity excess over the threshold state. In dimensionless terms

$$G_{cg} = 0.338\,(F_{cg} - 0.177)^{1.5} \qquad \ldots\ldots(59)$$

7. Sediment transport, especially in the lower range, is very sensitive to stream velocity. Accurate predictions of transport could only be obtained through accurate measurement of velocity, as well as of the other independent variables.

8. Previously published data* from flume tests of fine and medium sands broadly confirm the hypotheses on which the analyses were based. The transition coefficient, n, varies as predicted from $n = 0$ at $D_{gr} = 35$ to $n = 1.0$ at $D_{gr} = 7$.

9. The extent of the transition zone seems to be less than implied by previous studies, e.g. the conventional Shields curve. For sand, it extends from $D_{50} = 1.4$ mm to $D_{50} = 0.3$ mm.

10. The dimensionless coefficient A is a function of F_{gr}, but comparison with the Shields curve shows that extrapolation of an equation for well-established motion back to zero transport does not agree with the threshold condition on a plane bed.

11. A similar form of equation to that which holds for coarse sediment also applies to the transition range, but there is evidence of departure in the fine zone.

12. The sediment function tentatively put forward is defined by equations 49 to 57. It is illustrated in figures 9 and 10.

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* See footnote on page 31

13. The present equations are convenient both as a framework for analysing new data from laboratory and field, and also for design purposes, because they provide a very direct and straight-forward method of computation.

It is stressed that this report is an interim account* of a continuing programme of study.

ACKNOWLEDGEMENTS

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* The sediment transport rates quoted in ref 6 are apparently in terms of immersed weight whereas in the above analysis they were taken to be dry weight. The values of G_{gr} deduced from Williams data as plotted in figs 3, 4, 6 and 7 are thus in error: the correction factor is approximately 1.6. The next report on the study will correct this error in the analysis of a proportion of the data.

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NOMENCLATURE

		Dimensions
A	value of F_{gr} at nominal initial motion	-
D, D_{50}	median sediment diameter	L
E	sediment transport efficiency	-
e	error	-
F	sediment mobility number	-
Fr	Froude number	-
f	A function of ...	-
G	dimensionless sediment transport rate	-
g	acceleration due to gravity	LT^{-2}
g_b	sediment load dry weight per unit width per unit time	MT^{-3}
i	hydraulic gradient	-
k_s	sand roughness	L
M	number of tests	-
m	hydraulic mean depth	L
n	transition exponent depending on sediment size	-
p	packing factor	-
Re	Reynolds number	-
S	standard deviation	-
s	specific gravity of solids	$ML^{-2}T^{-2}$
V	mean velocity	LT^{-1}
v, v_*	shear velocity	LT^{-1}
w	fall velocity of sediment particles	LT^{-1}
X	sediment transport, mass flux per unit mass flow rate	-
y	depth of flow	L
α	numerical constant in rough-turbulent equation	-
β	a coefficient in transport function	-

ϕ	angle of friction	-
ρ	mass density	ML^{-3}
τ	shear stress	$ML^{-1}T^{-2}$
ν	kinematic viscosity of fluid	$L^{2}T^{-1}$

Suffixes:

b	denotes a bed process
m	measured
o	nominal initial motion
p	predicted
s	denotes a suspension process
cg	denotes coarse grain situation
fg	denotes a fine grain situation
gr	denotes a dimensionless parameter for any grain size of sediment

TABLES

TABLE 1

Results of analysis for coarse sand

Source	D_{50} mm	D_{gr}	$A = \frac{V}{\sqrt{32g(s-1)D}}$ at nominal initial motion	
Williams	1.35	34	0.168	Average value 0.177
U.S.W.E.S., sand 9	4.08	102	0.170	
Liu, sand II	3.4	96	0.193	
Liu, sand III	2.3	66	0.188	
Liu, sand IV	1.4	38	0.177	
Liu, sand VI*	1.8	50	0.173	

* mixture

TABLE 2

Results of analysis for fine and transition-size sands

Data Source	D_{50} mm	D_{gr}	No. of tests M	Plotted in fig.	Optimised n	Optimised A	Std Deviation $S = \sqrt{\left(\frac{\Sigma e^2}{M}\right)}$	% scatter in F_{gr}
Guy et al, ref 7								
Table 2	0.19	4.47	26	6 (a)	1.022	0.445	0.081	21
Table 3	0.27	6.35	14	6 (b)	0.971	0.436	0.052	13
Table 4	0.28	6.60	25	6 (b)	1.012	0.467	0.043	10
Table 5	0.45	10.59	25	6 (d)	0.590	0.241	0.053	13
Table 6	0.93	23.02	28	6 (f)	0.478	0.217	0.035	8
Table 7	0.32	7.53	16	6 (c)	1.119	0.541	0.032	8
Table 8	0.33	7.77	6	6 (c)	0.935	0.438	0.024	6
Table 9	0.33*	7.77	10	6 (c)	0.914	0.338	0.047	11
Table 10	0.47	11.04	33	6 (d)	0.782	0.327	0.036	9
Table 11	0.54	12.71	18	6 (e)	0.600	0.246	0.048	12
Kennedy and Brooks, ref 8	0.14	3.31	19	6 (a)	1.152	0.494	0.031	7
Williams, ref 6	1.35	31.8	24	3	0.072	0.178	0.009	2

* graded

FIGURES

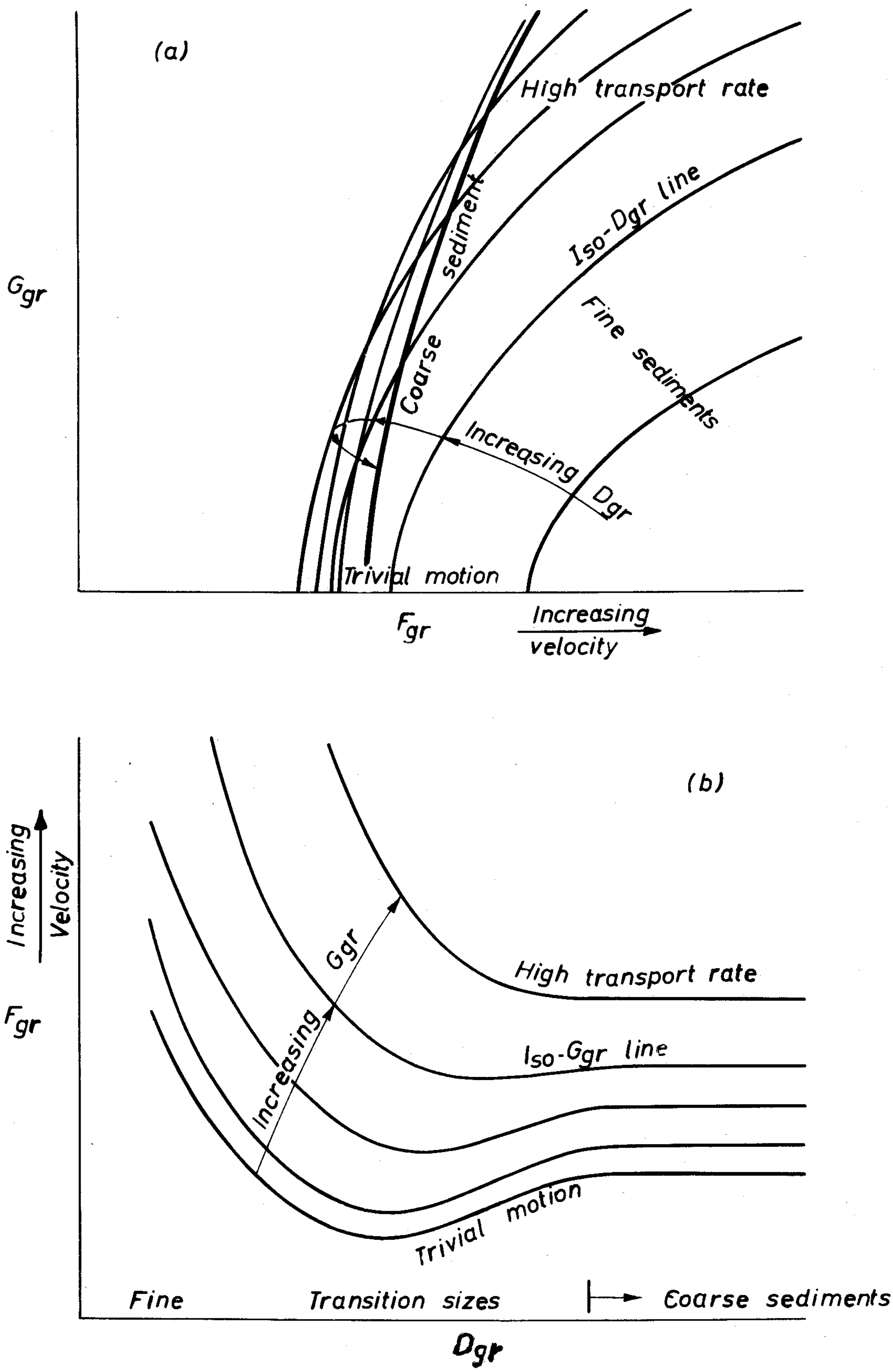


FIG 1. ANTICIPATED GENERAL FORM OF TRANSPORT FUNCTION

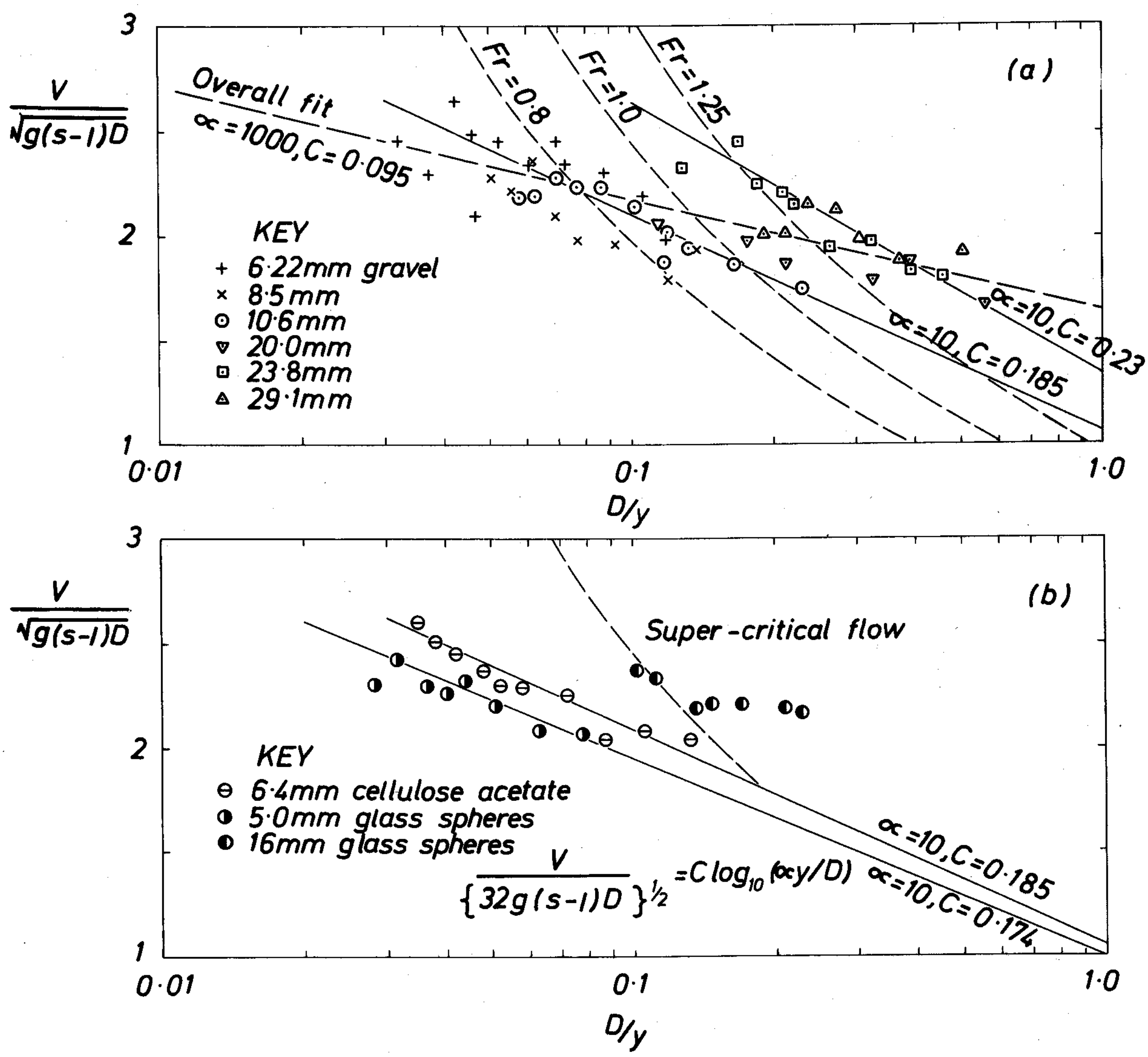


FIG 2. EVALUATION OF COEFFICIENT ∝ FROM INITIAL MOTION OF COARSE SEDIMENTS

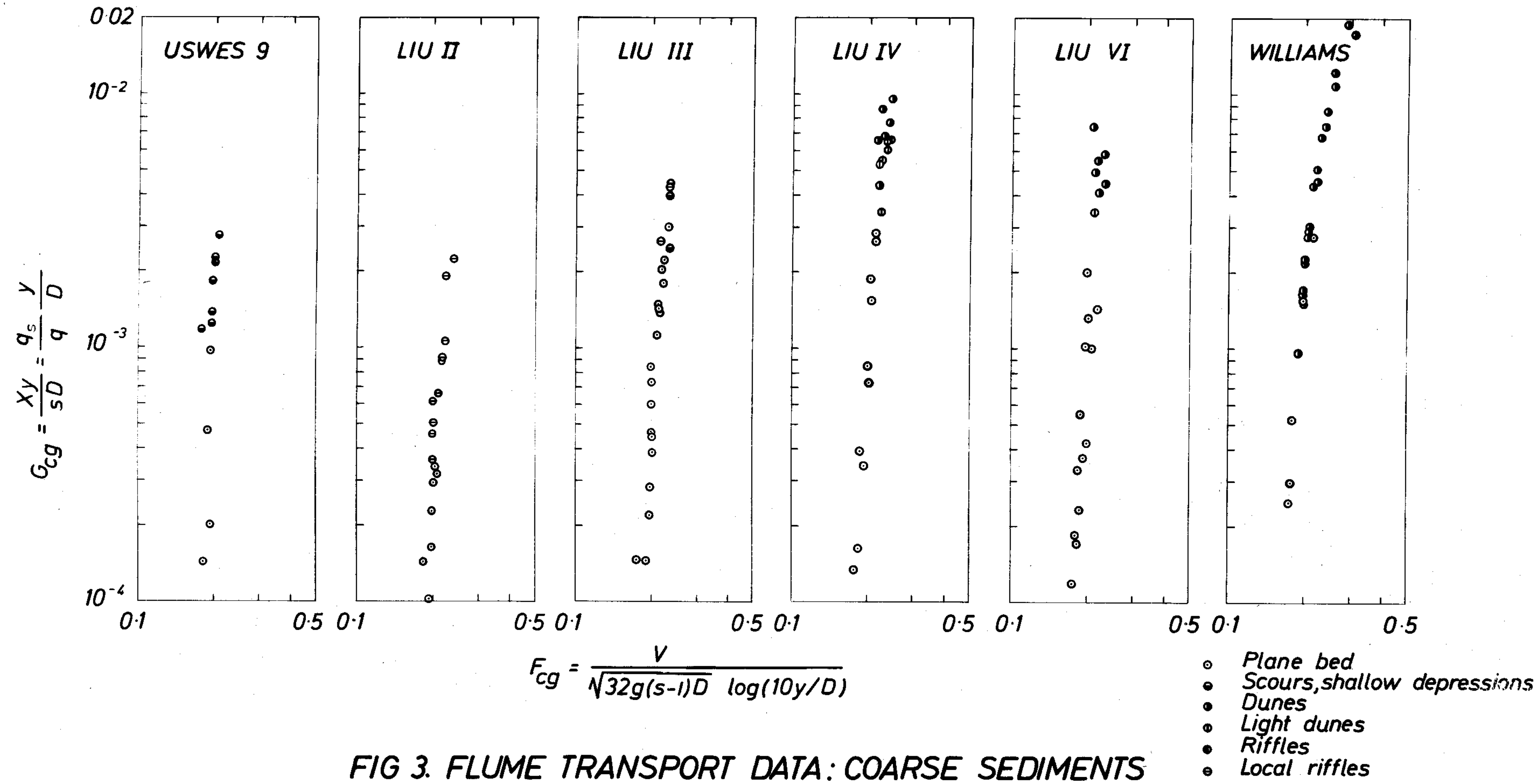


FIG 3. FLUME TRANSPORT DATA: COARSE SEDIMENTS

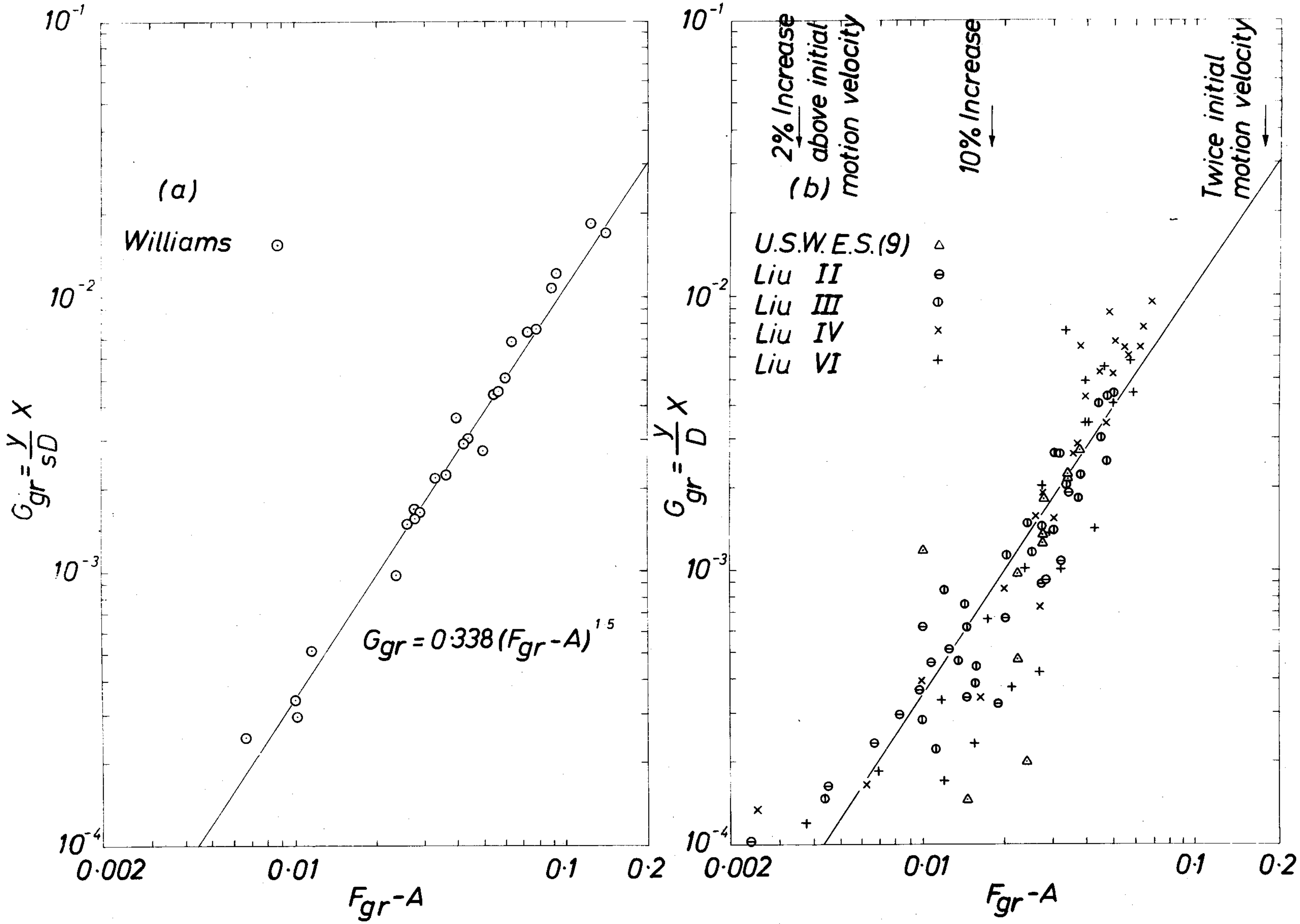


FIG 4. TRANSPORT DATA AND EQUATION : COARSE SEDIMENT

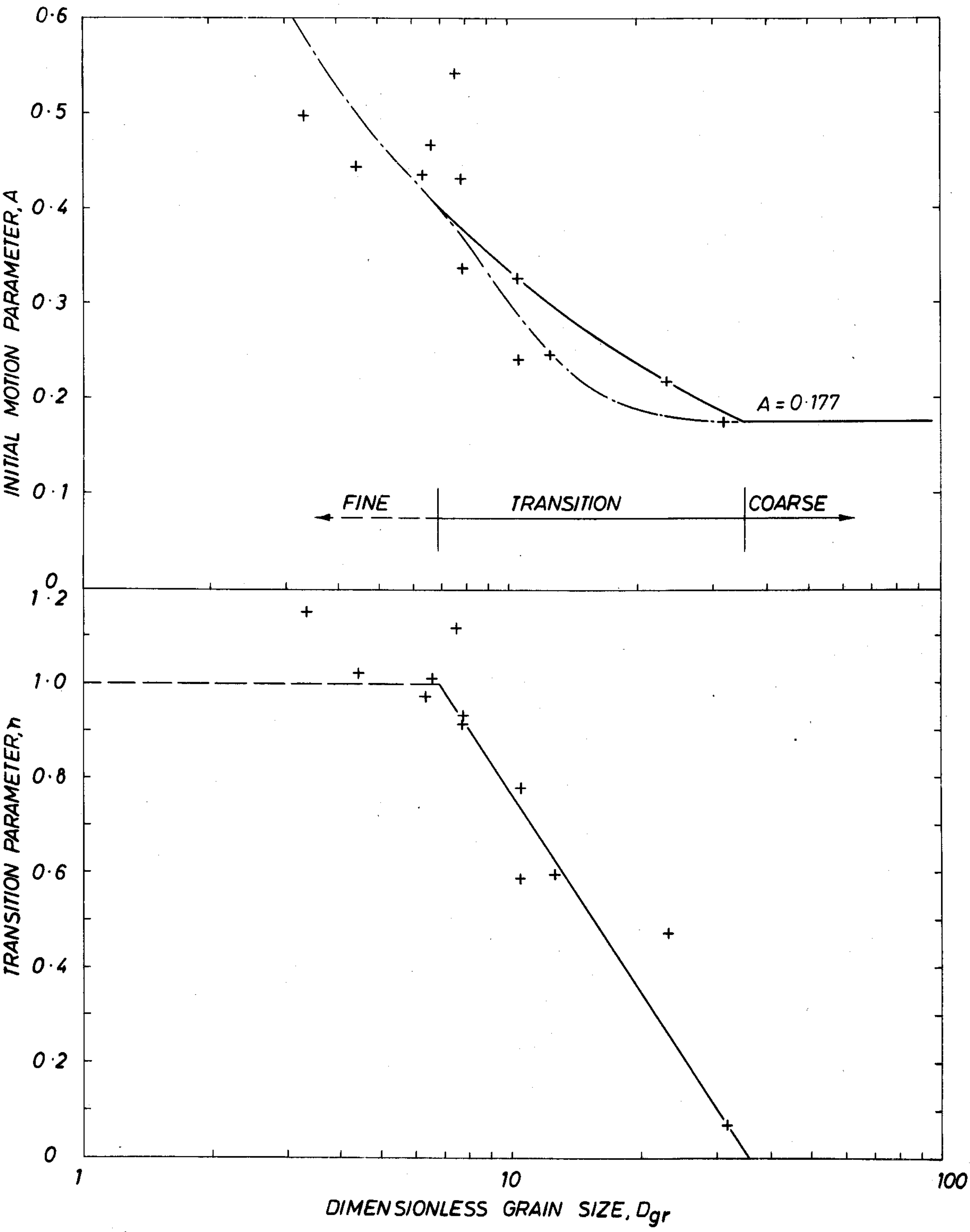


FIG 5. VARIATION OF n AND A WITH D_{gr}, TRANSITION ZONE

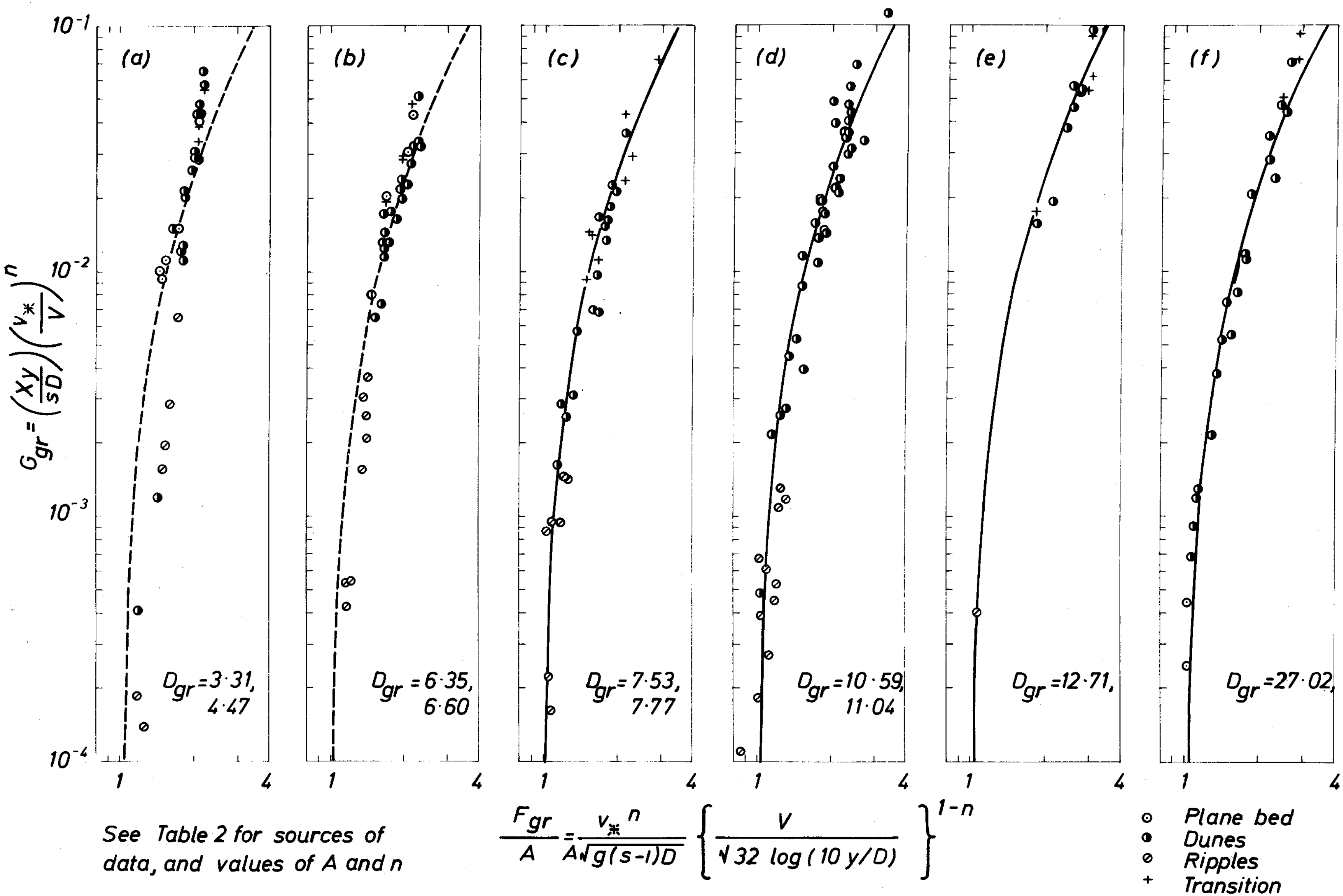
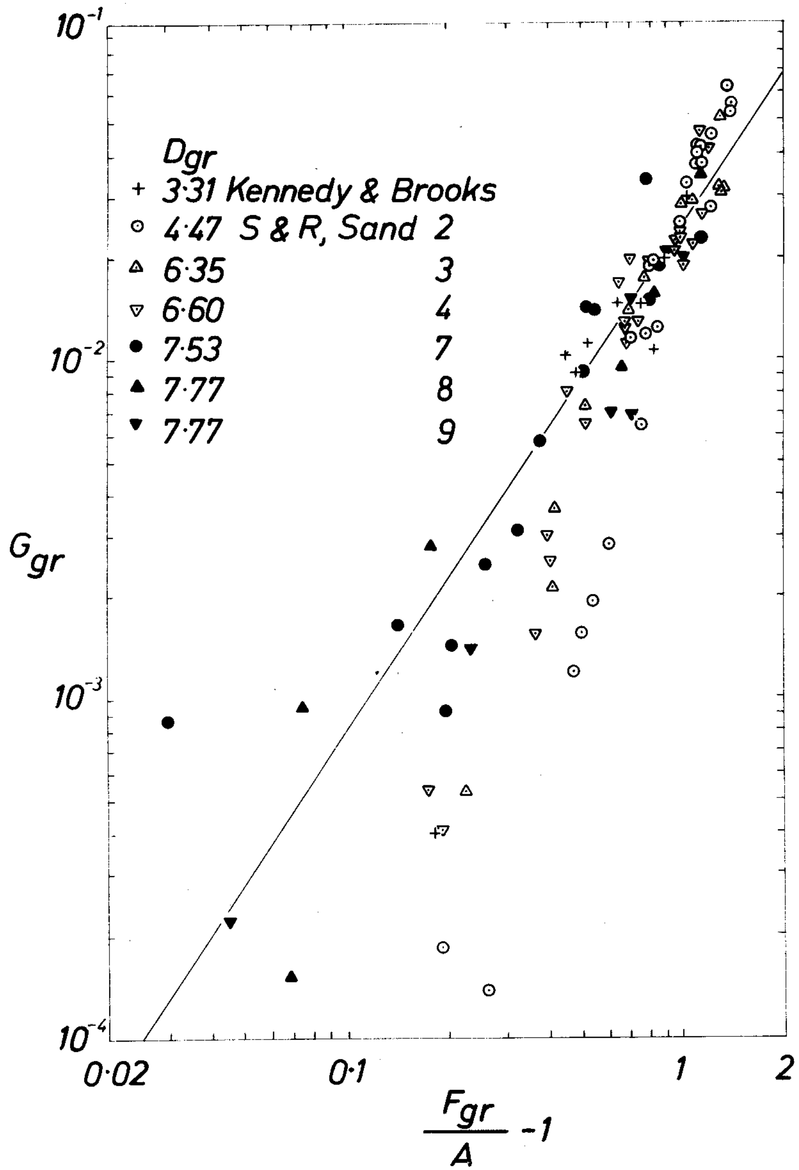


FIG 6. FLUME TRANSPORT DATA : TRANSITION AND FINE SEDIMENTS



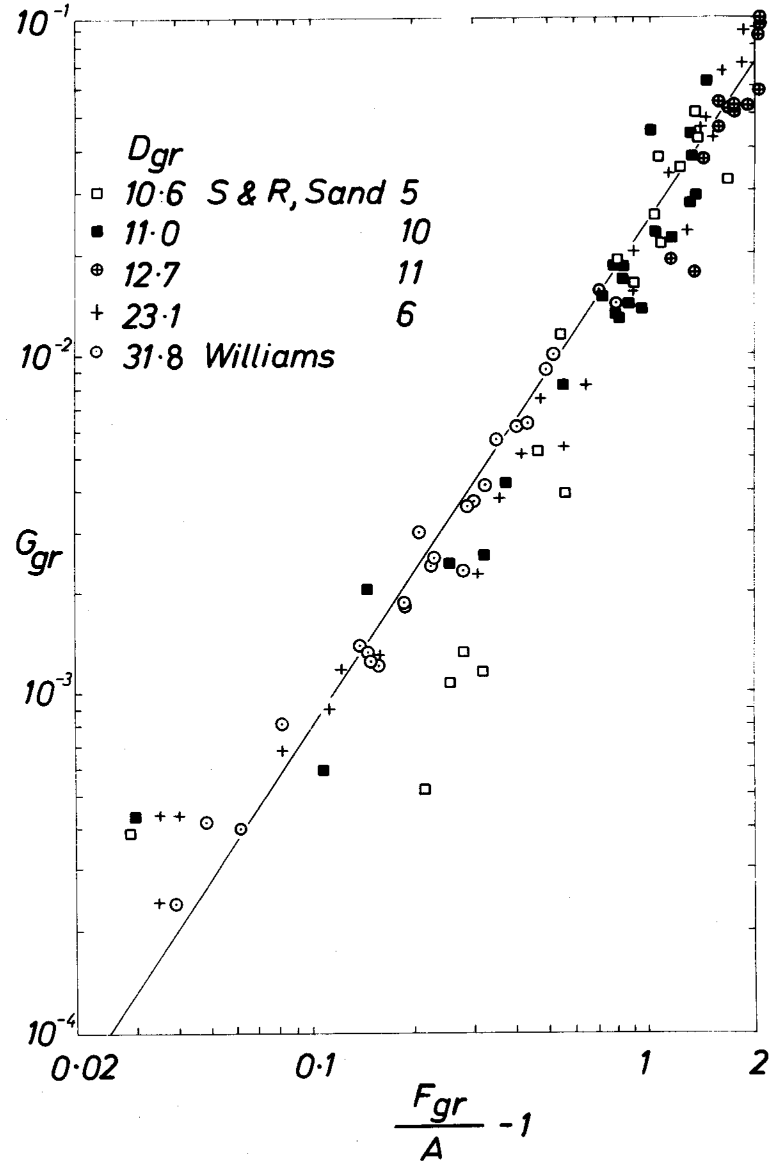


FIG 7 TRANSPORT DATA AND EQUATION : TRANSITION SEDIMENTS

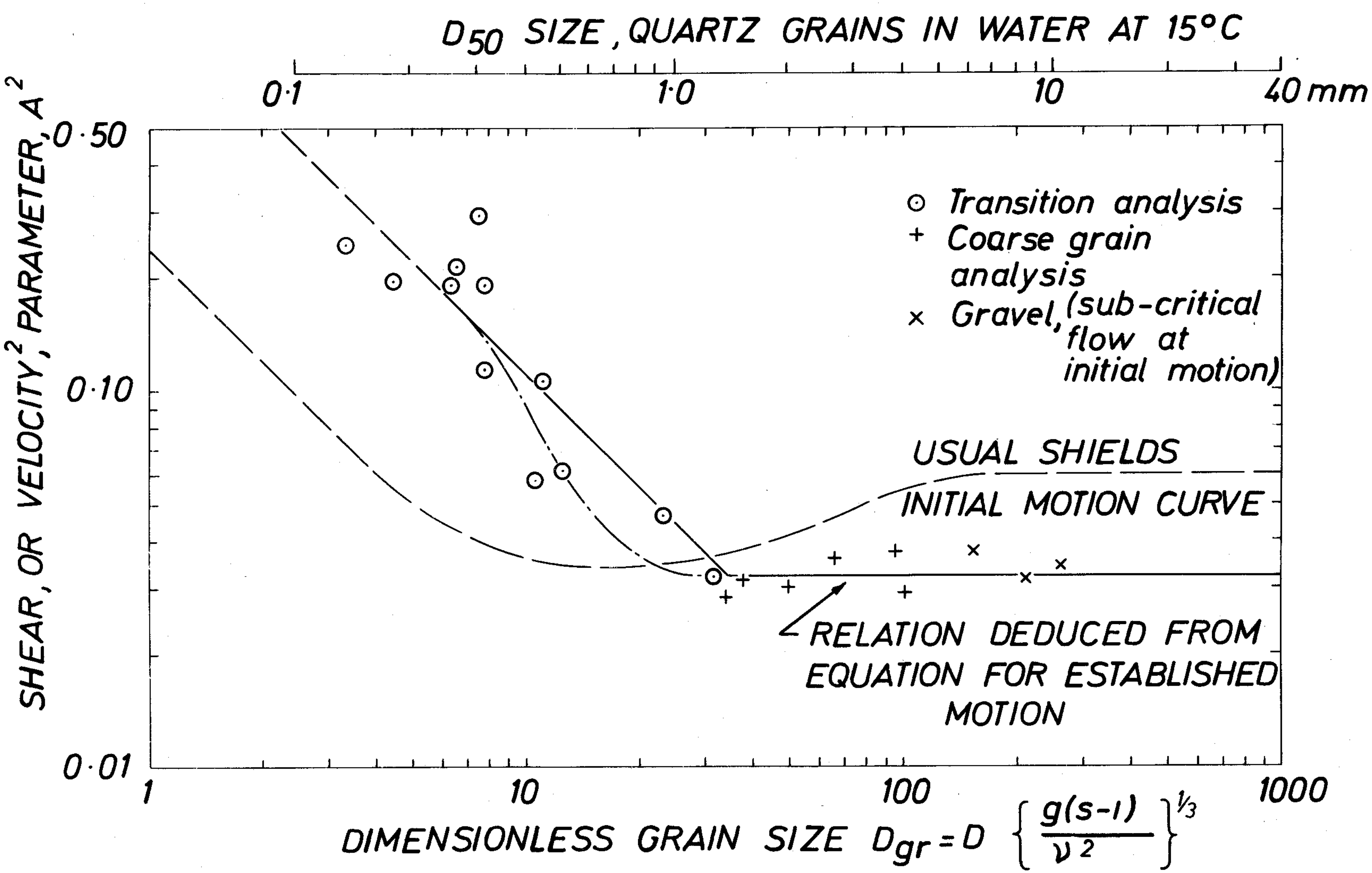


FIG 8. NOMINAL INITIAL MOTION CONDITION DEDUCED FROM EQUATION FOR ESTABLISHED MOTION

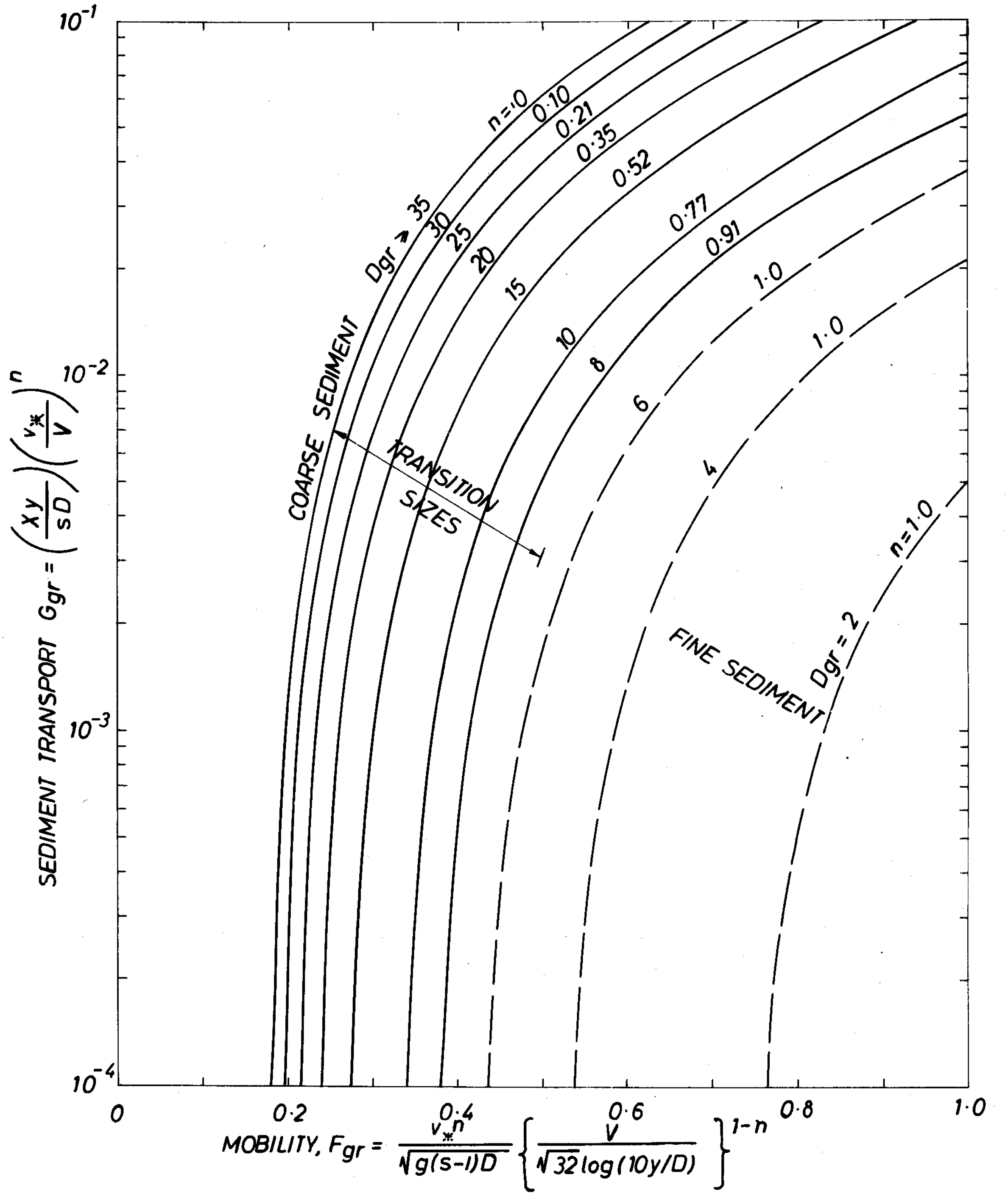


FIG 9. TRANSPORT FUNCTION = G_{gr}, F_{gr} FOR ISO - D_{gr}

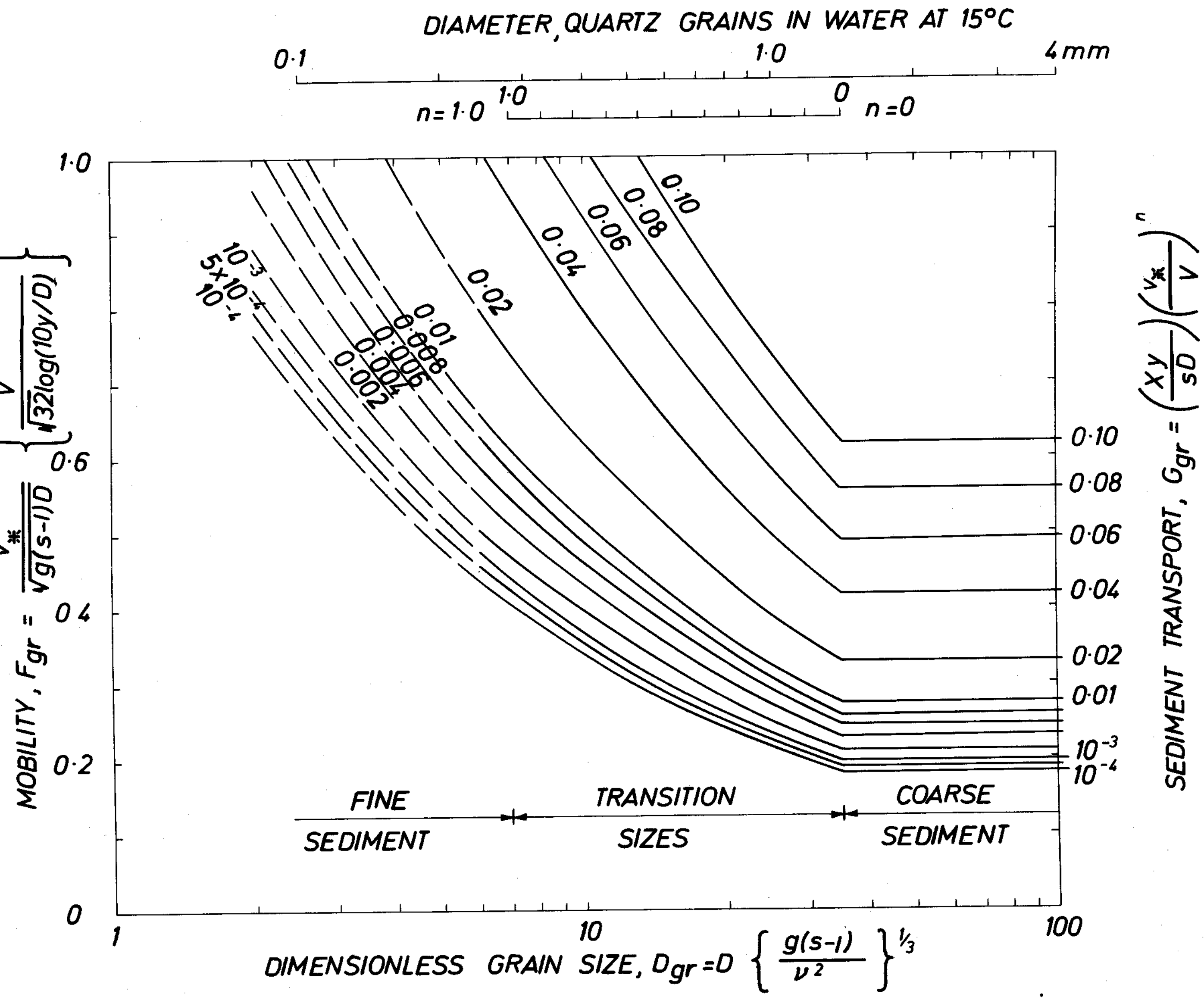


FIG 10. TRANSPORT FUNCTION : F_{gr}, D_{gr} FOR ISO - G_{gr}