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Representing the effects of currents in wave
disturbance models

- A review of computational modelling methods

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ABSTRACT

This report describes the outcome of a review of computational modelling
methods for representing currents in wave disturbance models. The purpose
of the review is to allow the most appropriate method to be selected prior
to implementation.

The review essentially comprises three parts. In the first of these the
available mathematical representations of the physical processes are
considered. Having identified these the different methods of numerical
solution for the various equations sets are summarised. The final part
describes the data sets which are available for validation. At all stages a
discussion of the various aspects is given, and the overall outcome of these
discussions is presented in the conclusions.

NOMENCLATURE

A   wave action

A   amplitude of water surface fluctuation

a   C Cg

C   wave celerity

Cg   group velocity

g   gravitational acceleration

h'   depth

K   wave number (modelling parameter if currents included)

m   ($=k/ko$) index of refraction

n   ($C=Cg/c$)

$\underline{n}$ unit normal to water surface

Q   flow rate

r_f reduction factor

t   time

$\underline{u}$ total particle velocity

$\bar{\underline{u}}$ steady/ambient velocity

$\underline{u}'$ unsteady velocity

$\underline{x}$ two dimensional position vector

z   vertical position vector

∇ horizontal gradient operator

∇_3 three dimensional gradient operator

η water surface elevation

$\bar{\eta}$ mean vertical position of water surface

η' variable vertical position of water surface

NOMENCLATURE   (Cont'd)

Φ Φ (x,z,t) complex velocity potential

Φ Φ(x,t) complex velocity potential representing Φ at z=o

ϕ ϕ(x) complex velocity potential

φ scaled velocity potential

σ relative angular wave frequency

ω angular wave frequency (absolute if currents included)

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1   INTRODUCTION

1.1 Background

At present, most computational models used in the
determination of wave conditions in harbours neglect
the effect of currents on the waves. This
approximation is justifiable at some sites but in many
situations significant currents exist around the
harbour which modify the wave climate in that
locality.

The presence of a current will change the wave speed,
and thus alter the dependence between the wavelength
and period. In travelling between areas with
different currents the wave speed and length alter,
leading to refraction by currents, and a resulting
modification in wave height and direction. In some
cases this may induce breaking. These factors will
lead to different wave effects within the harbour, and
consequent changes in the movement characteristics of
moored vessels.

For harbours which are entered through a dredged
approach channel, changes in wave height and direction
due to current refraction will have a significant
effect on both siltation and ship manoeuvrability. In
particular, a more accurate assessment of the
hydrodynamics including both the effects of waves and
currents, will allow much better estimates of
siltation rates in dredged channels to be made.

There are a wide variety of mathematical techniques
which can be used to solve the equations representing
the effect of currents in wave disturbance models.
Thus, before embarking on a development of a model it
is necessary to review these techniques, so that the
most appropriate to the physical problem can be

selected. In addition, possible sources of data for verification of numerical models including currents needed to be identified. Both of these aspects are discussed in this report. Once the method for representing current effects has been selected, the next stage is to implement and validate it. This will be done during the subsequent phases of this project.

1.2 General considerations

This report describes and discusses methods available for representing mathematically wave disturbance under the influence of currents within and in the approaches to harbours.

Numerical models are based on the solutions of mathematical statements which characterise the features of the physical problem. The accuracy of the prediction will depend on how well the governing equations represent these mechanisms, and the accuracy of the numerical approximation to those equations. Therefore within the bounds of a desired level of accuracy and available computational power we have two main problems: defining a mathematical statement which best represents the physics, and secondly developing a numerical method which solves the statement or its equivalent as accurately as possible. Furthermore, in order that the numerical models are economically viable our problem involves a compromise between accuracy and computational efficiency. Ideally we require a model which solves the equations rapidly and which satisfies our accuracy requirements.

The methods most frequently used at HR in wave disturbance studies use ray tracking techniques. These provide an efficient method for preliminary assessment of harbour layout. Ray models can be modified to include the effects of currents on wave

refraction in a relatively straightforward way.
However, ray models have one significant limitation,
and that is that they do not include diffraction
effects in the governing equations. Diffraction by
breakwaters can be represented in this type of model
by explicitly including its effects. However,
diffraction by sharp discontinuities in the bed such
as those which can occur at shoals or dredged channels
will not be modelled. Thus whilst a ray model
modified to include current effects will give a
reasonable representation of physical process, a much
more accurate description will be provided by using a
technique which includes bed diffraction in the
governing equations.

An alternative approach to modelling wave disturbance
is to use a mathematical representation based on the
mild slope equation. This includes both refraction
and diffraction effects in its governing assumption,
and can be modified to include current effects. This
equation, or its derivatives, are most frequently
solved using either finite difference or finite
element techniques.

1.3   Outline of report

As a starting point a general description of the
effects of currents in wave disturbance models is
given. This includes a description of the physical
problem and the mathematical representation, and is
presented in Chapter 2. The solution methods which
are available are described in Chapter 3, together
with a discussion of which is the most appropriate for
the applications considered here. In Chapter 4 the
data available for verification of wave/current models
in harbour applications is reviewed. The conclusions
and recommendations resulting from this study are
given in the final chapter.

2 THE EFFECTS OF CURRENTS IN WAVE DISTURBANCE MODELS

2.1 Representation of the physical problem

In the approaches to a harbour wave propagation is influenced by refraction and shoaling due to depth variation, diffraction by sharp discontinuities at the seabed, shallower regions inducing wave breaking and seabed friction. Within the harbour, diffraction by breakwaters is also of importance, as are reflections from the internal structures. In an area where currents are strong these will also have a significant effect on wave propagation.

The physics of wave refraction, shoaling and diffraction is well understood and can be modelled mathematically with reasonable accuracy. The effects of bed friction and wave breaking are less well understood, and as a result mathematical models often rely on an empirical representation of their effect on wave propagation. The influence of large scale currents on wave activity is also reasonably well researched, but it is only in recent times that mathematical models have begun to be developed to include their effects on waves.

2.2 Mathematical representations

As a starting point we need to review the methods available to represent the effects of currents in wave disturbance models. Two basic representations, referred to as ray methods and the mild slope equation, will be considered. A brief account of each of these is given here, a detailed description of the mild slope equation will follow in later sections.

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Much of the relevant mathematical theory describing
the process of water wave refraction, reflection and
diffraction by impermeable barriers are analogous to
the theory of light. From this derives ray theory
which is in frequent use in many mathematical models
of wave activity (see for example Ref 1). This
approach can be modified to include the effect of
large scale currents, but this has so far only been
carried out for coastal wave refraction models, ie
models which do not include diffraction by breakwaters
or reflections from structures. An account of this
modification is given in Reference 2.

It is therefore possible to extend existing ray
models of wave disturbance for harbour applications to
include current effects. This would be a relatively
straightforward process, and would provide a model
which would be capable of giving a preliminary
estimate of the effects of currents on wave
disturbance in harbour studies. However, as discussed
briefly in the preceeding chapter, ray methods do not
include in their governing equations diffraction
effects. Diffraction by breakwaters can be included
by other means, but diffraction caused by
discontinuities at the seabed will not be adequately
represented. In many harbour studies, for example
where access is by a dredged channel, this process
will be significant.

Thus, before increasing the complexity of the physics
of a model by including currents, the accuracy of the
representation of wave effects can be improved. This
can be achieved by using the mild slope equation as
the basis for a wave disturbance model. The mild
slope equation incorporates diffraction effects in its
formulation, and can also be extended to include
current effects. A detailed derivation of three
different forms of the equation follows. For clarity

the mathematical derivations without currents will be
described first, and then the modifications required
to include currents.

2.3 The mild slope equation

2.3.1 Elliptic form

The mild slope equation for which the derivation is
credited to Berkhoff (Ref 3) is given by

$$\nabla(C\ C_g\ \nabla\phi) + \frac{C_g}{C}\ \omega^2\ \phi = 0 \qquad\qquad (1)$$

where ∇ is the horizontal gradient operator

C is the wave celerity ($= \omega/k$)

C_g is the group velocity $d\omega/dk$

$\phi = \phi(\underline{x})$ a complex velocity potential at the
mean free surface, and by introducing a time
harmonic motion

$\Phi(\underline{x},t) = Re\ \{e^{-i\omega t}\ \phi(\underline{x})\}$ is related to the
potential $\Phi\ (\underline{x},z,t)$ by

$$\Phi\ (\underline{x},z,t) = \frac{\cosh\ k(z+h)}{\cosh\ kh}\ \Phi(\underline{x},t)$$

k is the wavenumber obeying the dispersion
relation $\omega^2 = gk\ \tanh\ kh$

and ω is the angular frequency.

This equation is accepted as a suitable equation for
modelling refraction and diffraction processes
provided the bed slope is as implied mild, ie slowly
varying. It is derived using linear wave theory under
the assumptions that the flow field is incompressible,
irrotational and homogeneous. Alternative derivations
by Smith and Sprinks (Ref 4), Lozano and Meyer (Ref 5)
and Behrendt and Jonsson (Ref 6) result in the same
equation (1).

The mild slope equation is elliptic in form. This means that it is a boundary value problem, and its solution is independent of any initial conditions. One result of this is that it is computationally time consuming to solve numerically using a finite difference or finite element scheme. Therefore approximations to the mild slope equation have been sought which are more efficient to solve. It is possible to derive both hyperbolic and parabolic approximations to the equation. These are initial value problems which are computationally more efficient when solving numerically. However, the approximations required to derive them do impose limitations on the physics originally represented by the mildslope equation. To clarify this point an outline of the derivation and assumptions employed in the hyperbolic and parabolic forms are given in the following sections.

2.3.2 Hyperbolic approximation

Almost simultaneously, but independently of the derivation of the mild slope equation in Reference 3, Ito and Tanimoto (Ref 7) derived a set of linear hyperbolic equations to model refraction and diffraction processes in coastal regions. Later Copeland (Ref 8) and Watanabe et al (Ref 9) independently of each other produced similar equations. In Reference 8 a transient form of the mild slope equation is derived which is subsequently represented by a pair of first order equations. Whereas, the derivation in Reference 9 is based on substituting a relation between velocity potential and surface elevation into the mild slope equation.

By expressing the water surface elevation η as

$$\eta = A(x,y) e^{-i(\chi - \omega t)}$$

where A is the amplitude of the water surface
　　　fluctuation

and　　χ is the phase angle

into equation (1) and equating real and imaginary
parts, the equations equivalent to (1) given in
Reference 9 are written as

$$\frac{\partial Q}{\partial t} + \frac{C^2}{n} \nabla(n\ \eta) = 0$$

$$\frac{\partial \eta}{\partial t} + \nabla.Q = 0$$

(2)

where $n = \dfrac{C_g}{C}$, η is the water surface elevation and Q,
the dummy variable, is the flow rate defined as a
vertically integrated function of particle velocity.
The equations can represent diffraction, refraction
and reflections under the assumptions made in their
derivation.

By creating a hyperbolic form from the original
equation the mild slope problem has been embedded in a
larger space (x,y,t). This appears to be an
unnecessary complication as the time dependence,
e^{-iwt}, is known in advance. Therefore timestepping
will produce only a phase change. If it does not then
there is a basic inconsistency in the derivation.

This point has been discussed by Saville (Ref 24), and
explored further by Madsen and Larsen (Ref 25). They
make the observation that the time stepping is
actually only an iteration towards the steady state,
and that only the steady state solution is a solution
to the mild slope equation. This accounts in part in
the difficulties which are known to occur in getting
the hyperbolic form to converge to the steady state.

A difficulty which needs to be resolved before the method can be used reliably in practice.

The solution offered by Madsen and Larsen is to extract the time harmonic term out of the equations set (2), and reformulate the hyperbolic form. This seems to offer a partial resolution to the difficulties described above. However, the method of solution and achieving convergence relies on a complex variable time stepping numerical scheme which does not offer as efficient a method of solution as anticipated.

A possibility not yet explored in the literature is to formulate a hyperbolic approximation with an iteration parameter which is not time dependent. If suitably selected this would offer a form of the equations which could be solved using a more rapid solution technique.

2.3.3 Parabolic approximation

The time taken to solve the elliptic mild slope equation computationally, and the mathematical uncertainties of the hyperbolic form, means that attention should be given to the parabolic approximation. This will be computationally efficient to solve and mathematically more rigorous in its derivation. However, this is achieved at the expense of accuracy in the representation of physical problem. That is, whilst refraction and diffraction are represented in the parabolic approximation, reflections are not.

The derivation of the parabolic approximation is given by Radder in Reference 10. Here Helmholtz's equation is considered in a wave field consisting of forward and backward propagation fields. The derivation

assumes the backward or reflected wavefield is
negligible compared to the forward or transmitted
field. Radder's parabolic approximation to the mild
slope equation is given by

$$\frac{\partial^2 \varphi}{\partial y^2} + 2ik_o m \frac{\partial \varphi}{\partial x} + \left(2k_o^2 m^2 + ik_o \frac{\partial m}{\partial x}\right)\varphi = 0 \qquad (3)$$

where k_o denotes a constant wavenumber, and $m = k/k_o$
the index of refraction, and φ a scaling factor
defined by $\varphi = \phi (CCg)^{\frac{1}{2}}$

2.4 Modification of the mild slope equation to include current effects

2.4.1 Elliptic form

The numerical solution of wave current interaction
problems involving large scale currents using
techniques other than traditional ray methods are
mostly based on, or are approximations to, the
mathematical formulation developed by Booij in
Reference 11. The derivation given there is not
entirely rigorous mathematically, and corrections were
put forward by Kirby (Ref 12). However the basic
principles are sound, and a brief account of the
derivation is given here.

The introduction of currents into the model implies
the total particle velocity:

$$\underline{u} = \bar{\underline{u}} + \underline{u}' \qquad (4)$$

where $\bar{\underline{u}}$ is the steady velocity due to currents and
$\underline{u}'$ is an unsteady velocity with mean zero.

Since the motion due to waves is assumed irrotational the velocity $\underline{u}'$ can be represented by a potential Φ by:

$$\underline{u}' = \underline{\nabla}_3 \Phi \tag{5}$$

Then assuming the lower surface, or bed, at:

$$z = -h'(\underline{x}) \tag{6}$$

where $\underline{x} = \underline{x}(x,y)$ is the two dimensional position vector is rigid and impermeable, the boundary can be represented by:

$$\underline{n} \cdot \underline{u}' = \underline{n} \cdot \underline{\nabla}_3 \Phi = 0$$

where $\underline{n}$ is the unit normal to the surface.

At the upper surface defined by

$$z = \eta(\underline{x},t) = \bar{\eta}(\underline{x}) + \eta'(\underline{x},t) \tag{8}$$

where $\bar{\eta}(\underline{x})$ is the mean vertical position of the free surface, and $\eta'(\underline{x},t)$ is the variable part, two conditions must be satisfied:

$$\frac{\partial \eta'}{\partial t} + \bar{\underline{u}} \cdot \underline{\nabla} \eta' - \frac{\partial \Phi}{\partial z} = 0 \text{ on } z = \bar{\eta}(\underline{x}) \tag{9}$$

$$\frac{\partial \Phi}{\partial t} + \underline{\nabla} \cdot (\underline{u}\, \Phi) + g\eta = 0 \text{ on } z = \bar{\eta}(\underline{x}) \tag{10}$$

which can be combined into

$$\left(\frac{\partial}{\partial t} + \bar{\underline{u}} \cdot \underline{\nabla}\right) \left(\frac{\partial \Phi}{\partial t} + \underline{\nabla} \cdot (\bar{\underline{u}}\, \Phi)\right) + g\frac{\partial \Phi}{\partial z} = 0 \text{ as } z = \bar{\eta}(\underline{x}) \tag{11}$$

For time harmonic linear waves propagating in a region of uniform depth and a homogeneous current field the potential Φ can be expressed in terms of the two dimensional potential ϕ as

$$\Phi(\underline{x},z,t) = \text{Re} \{e^{-i\omega_o t} f(z) \phi(\underline{x})\} \tag{12}$$

$$\text{and } f(z) = \frac{\cosh\{k(h'+z)\}}{\cosh\{k(h'+\bar{\eta})\}} \tag{13}$$

for which the wave number k satisfies

$$\sigma^2 = g\,k\,\tanh(kh) \tag{14}$$

$$\sigma = \omega - \underline{k}.\underline{\bar{u}} \tag{15}$$

$$\text{and } h = h' + \bar{\eta} \tag{16}$$

Here σ is the relative angular wave frequency and ω is the absolute angular wave frequency.

k is a vectoral quantity whose direction is related to direction of propagation of waves. k is known if this direction of propogation is known in advance.

Substituting (12) and (13) into equation (11) gives

$$(\frac{\partial}{\partial t} + \underline{\bar{u}}.\nabla)(\frac{\partial\Phi}{\partial t} + \nabla.(\underline{\bar{u}}\Phi) - \nabla.(a\,\nabla\Phi) + (\sigma^2 - k^2 a)\Phi = 0 \tag{17}$$

where $a = C\,C_g$
and $\Phi(\underline{x},z,t) = f(z,h)\Phi(\underline{x},t)$ \hfill (18)

An elliptic version of (17) for purely periodic waves is given by

$$-i\omega \ (\bar{u}. \ \nabla\phi + \nabla(\bar{u}\phi))+ (\bar{u}.\nabla)\nabla. \ (\bar{u}\phi)$$

$$-\nabla.(a \ \nabla\phi) + (\sigma^2 + \omega^2 - k^2a)\phi = 0 \qquad\qquad (19)$$

where k can now be regarded as a modelling parameter.

Equation (17) yields the dispersion relation

$$(\omega - \underline{\kappa}.\bar{u})^2 + - a \ \kappa^2- (\sigma^2 - a \ k^2) = 0 \qquad\qquad (20)$$

and if k obeys (15) and (16) $\underline{\kappa} = \underline{k}$.

In most cases, an approximation for $\underline{k}$ is necessary since crossing and reflected waves exist making (15) and (16) void.

However the main direction of propagation is often known, and approximations to k can be used to determine k. At worst, the relation

$$\omega_o^2 = gk_o \ \tanh \ (k_oh) \qquad\qquad (21)$$

assuming zero mean velocity can be used with error of order $O(|\bar{u}|/C)$ (Ref 10). For propagating models the value of k can be determined more accurately in a step wise fashion dependent on the direction of propagation.

Kirby (Ref 12) detected an error in the derivation of dynamic free surface boundary condition and replaces (10) with:

$$\frac{D\phi}{Dt} + g\eta = 0 \qquad\qquad (22)$$

where the operator $\dfrac{D}{Dt} = \dfrac{\partial}{\partial t} + \bar{\underline{u}}.\nabla$

and gives the hyperbolic equation

$$\frac{D^2\Phi}{Dt^2} + (\nabla \cdot \bar{u}) \frac{D\Phi}{Dt} - \nabla(a \nabla\Phi) + (\sigma^2 - k^2 a)\Phi = 0 \qquad (23)$$

Kostense (Ref 13) introduces a time harmonic solution

$$\Phi(\underline{x},t) = \text{Re} \ (\phi(\underline{x})e^{-i\omega t}) \qquad (24)$$

which results in the elliptic equation:

$$\frac{\partial}{\partial \underline{x}} (a \frac{\partial \phi}{\partial \underline{x}}) + 2i \ \omega \ \bar{\underline{u}} \ \frac{\partial \phi}{\partial x} + (k^2 a - \sigma^2 + \omega^2 + i\omega\nabla \cdot \bar{\underline{u}})\phi$$

$$= - \sigma W\phi \qquad (25)$$

including dissipation terms.

A slightly different approximation to k to that in Reference 10 is given in Reference 13, and is described below.

Initially by setting $\bar{u} = 0$ or the direction of $\underline{k}$ equal to that of the incident wave direction an approximate solution ϕo is determined from (25) and the approximation to the wave number vector $\underline{k}$ is

$$\underline{\kappa}_0 = \text{Im} \ \{\frac{\nabla\phi o}{\phi o}\} \qquad (26)$$

The direction of $\underline{Ko}$ is used as an estimate for the direction of $\underline{K}$ in (15) such that

$$\sigma = \omega - [\frac{\bar{\kappa}_0}{\kappa_0} k] \cdot \underline{u} \qquad (27)$$

Then using (14) σ and k can be solved for and substituted into (25) to give a value of ϕ_1.

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Successive iterations of this procedure is applied
until a predetermined level of accuracy is met.
Usually 5 iterations of this procedure are necessary
according to Reference 13 to obtain reliable estimates
of $\nabla\phi$.

2.4.2 Hyperbolic form

Following the order of the previous section we now
consider formulations of the mild slope equations with
currents which result in linear hyperbolic transient
equation sets.

In Reference 14, a form of Kirby's equation given by

$$-g \left(\frac{\partial\eta}{\partial t} + \nabla(\bar{u}\eta)\right) - \nabla(a \nabla \Phi) + (\sigma^2 - k^2 a) \Phi = 0$$

is considered.

Despite the dynamic free surface condition

$$\frac{D\phi}{Dt} = -g \eta \tag{29}$$

Phillips has shown for current depth refraction it is
possible to put

$$\phi = -\frac{ig}{\sigma} \eta \tag{30}$$

and by substitution into the above equation the
dependence ϕ is removed. The relative frequency σ, is
related to absolute frequency ω and wave number k by
equations (14) and (15).

To find the magnitude and direction of $\underline{k}$ the
kinematical conservation equation of wave number

$$\underline{C}_g \cdot \nabla \underline{k} + \nabla \sigma + \nabla(\underline{k} \cdot \underline{\bar{u}}) = 0 \tag{31}$$

where $\underline{C}g$ is the group velocity vector is considered. However, equations (15) and (31) are only valid in cases where both diffraction and reflection effects are neglible. Thus the approximate formulation such as given in Reference 10 and previously discussed could be employed.

Substitution of (30) into Equation (28), eliminating Φ gives

$$-\left(\frac{\partial \eta}{\partial t} + \nabla \cdot (\underline{\bar{u}}\eta)\right) - \nabla\left(-i \ a \ \nabla \left(\frac{\eta}{\sigma}\right)\right) + \frac{(\sigma^2 - k^2 \ a \ )}{\omega\sigma}\frac{\partial \eta}{\partial t} = 0 \tag{32}$$

Combining terms and introducing

$$Q = -i \ a \ \nabla(\eta/\sigma) \tag{33}$$

where $Q = (Q_x, Q_y)$ gives

$$\nabla Q + \nabla(\underline{\bar{u}}\eta) + \lambda \frac{\partial \eta}{\partial t} = 0 \tag{34}$$

$$\frac{\partial Q}{\partial t} + \omega \ a \ \nabla(\eta/\sigma) = 0$$

Ohnaka et al (Ref 15) consider (23) and separate it into two equations expressed in terms of the surface elevation η, and the flow rate vector $\underline{Q}$.

Defining $\underline{Q} = \int_{-h}^{0} \nabla \ \phi \ dz \tag{35}$

and relating Φ to Φ by

$$\Phi \quad = \Phi(x, t) \ f(z) \tag{36}$$

where $f(z) = \dfrac{\cosh k(h+z)}{\cosh kh}$ (37)

gives

$$\underline{Q} = C^2 \, \underline{\nabla}\Phi / g \qquad (38)$$

Expressing the complex velocity potential Φ in terms of amplitude and phase as

$$\Phi = \phi(x) \, e^{i\chi} \qquad (39)$$

where $\chi = \underline{k}.\underline{X} - \omega t$ is the phase function, the surface elevation can be written as,

$$\eta = - \, \frac{1}{g} \frac{D\Phi}{Dt}$$

then ignoring changes in the amplitude of the velocity potential as compared with the change in the phase gives

$$\frac{D\phi}{Dt} = - \, i\sigma\phi \qquad (40)$$

Using this result and the expressions for Q Ohnaka proposed

$$\frac{\partial \underline{Q}}{\partial t} + \omega \quad C^2 \nabla(\eta/\sigma) = 0 \qquad (41)$$

and

$$m \, \frac{\partial \eta}{\partial t} + \underline{\nabla}.(\underline{\bar{u}}.\eta) + \underline{\nabla}(n.\underline{Q}) = 0 \qquad (42)$$

where $m = 1 + (\sigma/\omega)(n-1)$ and $n = C_g/C$, to be the time dependent mild slope equation extended to a wave and current co-existing field. It should be anticipated that the same difficulties discussed in 2.3.2 in achieving an accurate solution

for the waves only case will also apply to the wave
and current formulation.

2.4.3 Parabolic form

We now consider the parabolic model described by Booij
(Ref 11). The analysis in Reference 11, with no
current acting shows that a parabolic representation
will be computationally the most economic, and one
would expect this also to be true for the case
involving currents. However, as previously mentioned
this efficiency is marred by its inability to model
backwardly scattered waves.

A procedure followed in Reference 11 is to make an
exact split between the forward and backward
propagating wavefields. Assuming the waves propagate
in the s-direction, say, then it is reasonably
justified to put

$$\sigma \;=\; \omega \;-\; r_f \, k \, \bar{u}_s \qquad\qquad 0 \le r_f \le 1 \qquad\qquad (43)$$

where r_f is a reduction factor expressing the fact
that waves do not exactly follow the s-direction.
Then by manipulation of equation 16 it can be
presented in or at least comparable with the form

$$\frac{\partial}{\partial s} \left(\frac{1}{\gamma} \frac{\partial \phi_m}{\partial s} \right) + \gamma \, \phi_m = 0 \qquad\qquad (44)$$

which can be split exactly into two equations each
representing either the transmitted or reflected
field.

With M defined by

$$M\phi = (\omega^2 - \sigma^2 + i\omega \ \nabla \cdot \underline{u}) \ \phi + \frac{\partial}{\partial n} (a \frac{\partial \phi}{\partial n}) + 2i\omega \ \underline{u} \ \frac{\partial \phi}{\partial n}$$

(45)

the parabolic approximation to the mild slope equation given in Reference 11 is written as

$$(\frac{i\omega \ \bar{U}}{a} + \frac{\partial}{\partial s}) \ \{(ak)^{\frac{1}{2}} \phi + \frac{P_1}{k} (ak)^{-\frac{1}{2}} M\phi\}$$

$$-ik(ak)^{\frac{1}{2}} \phi - iP_2 (ak)^{-\frac{1}{2}} M\phi = 0 \qquad (46)$$

which differs only slightly from the equation used in the numerical model CREDIZ, (Ref 16) which is based on the equation:

$$(\frac{i\omega \bar{u}}{a} + \frac{\partial}{\partial s}) \ \{ \ (ak)^{\frac{1}{2}} \phi + \frac{P_1}{k} (ak)^{-\frac{1}{2}} (M\phi + i\omega W\phi) \ \}$$

(47)

$$- ik(ak)^{\frac{1}{2}} \phi + P_2 (ak)^{-\frac{1}{2}} (-iM\phi + \omega W\phi) = 0$$

$$P_1 = P_2 + \frac{1}{2} \qquad 0 \le P_1 \le \frac{1}{2} \qquad (P_{1_{opt}} = \frac{1}{4}) \qquad (48)$$

and which includes the energy dissipation terms

$$W\phi = (W_b + W_f + W_g)\phi \qquad (49)$$

where W_b represents wave breaking, W_f, bottom friction, and W_g, wave growth.

Arguments put forward by Kirby (Ref 12) show that (28) leads to the current conservation of wave action relation

$$\frac{\partial}{\partial t} (A) + \nabla \cdot (A(\ C_g + \bar{\underline{u}})) = 0 \qquad (50)$$

where A is the wave action.

The parabolisation in Reference 12 yields an alternative equation to that in Reference 11. Firstly, by considering the conservation law for the wave action A , defined by

$$A = \tfrac{1}{2}\rho g \ A^2/\sigma \tag{51}$$

where A is the wave amplitude, the potential Φ can be written as

$$i \ g^{-1} \ \Phi = R \ e^{i\chi}$$

where $R = A/\sigma$, $\omega = \dfrac{\partial \chi}{\partial t}$ and $\tag{52}$

R an χ are both real in expression (52). Substituting (52) into equation (23) and setting the imaginary part to zero gives:

$$\frac{\partial \sigma}{\partial t} R + 2\sigma \ \frac{\partial R}{\partial t} + 2\sigma \ \underline{u}.\nabla R + \nabla.(\sigma\underline{u})R + \nabla(k \ a \ )R$$

$$+ 2 \ a \ k. \ \nabla R = 0 \tag{53}$$

Since the processes of refraction and diffraction may turn the calculated wave away from the x direction R must be allowed to be complex. Substituting (52) into (23) and setting $\chi = \int^x \underline{k} \ dx - wt$ gives a complex version at (53).

$$i\{\frac{\partial \sigma}{\partial t} R + 2\sigma \ \frac{\partial R}{\partial t} + 2\sigma\underline{u}.\nabla \ R + \nabla \ .(\sigma\underline{u})R + \frac{\partial \ (\sigma \ C_g)}{\partial x}R$$

$$+ 2\sigma \ C_g (\frac{\partial R}{\partial x})\} - \frac{D^2R}{Dt^2} - (\nabla_h.\underline{u}) \ \frac{DR}{Dt} + \nabla \ .(a \ g\nabla \ R) = 0 \tag{54}$$

Then by neglecting time dependence and terms contain $\underline{u}^2$ and assuming that $\frac{\partial}{\partial x} \sim \partial(\frac{\partial}{\partial y})$ the parabolic

equation given in Reference 12 is written as

$$2ikA_x + 2ik \left(\frac{u_y}{C_g+u_x}\right)A_y + i\ \frac{k\sigma}{(C_g+u_x)}\ \{\frac{\partial}{\partial x}\left(\frac{C_g+u_x}{\sigma}\right) +$$

$$\frac{\partial}{\partial y}\left(\frac{\bar{u}}{\sigma}\right)\}A\ +\ \frac{k}{\sigma(C_g+u_x)}\ )\ (a\ A_y)_y = 0 \qquad\qquad (55)$$

2.5 Discussion

Three different forms of mathematical representation
of the mild slope equation have been presented. A set
of representations without current interaction
preceeded those developed to include the action on
waves of a current field. The representations have
been categorised into equation forms: elliptic,
hyperbolic and parabolic. In addition to the mild
slope equation some consideration was also given to
ray methods. It is also possible to include current
effects in the ray models of harbour wave
disturbance.

Where the effects of currents are not included the
best representation of the physical processes is
provided by the elliptic form of the mild slope
equation. The hyperbolic form is mathematically
uncertain with respect to determining the steady state
solution. The parabolic form does not allow wave
reflections to be included; this will be a significant
limitation in harbour wave disturbance studies. In
situations where seabed diffraction is not an
important mechanism, ray models will also provide an
accurate representation of the physical problems.

Inclusion of current effects in all of the models
described here allows them to retain their basic form.
Thus, the selection of the best type of model to
represent both wave and current effects can largely be

based on their accuracy of representation of the waves
only situation. The foregoing discussion indicates
that the best representation will therefore be
provided by the elliptic form of the mild slope
equation. Ray models will also provide a reasonably
good method for many physical problems. These
comments do not make any allowances for methods and
efficiency of the numerical solution. These will be
discussed further in the next chapter.

3 NUMERICAL SOLUTION METHODS FOR THE MILD SLOPE EQUATION

3.1 Introduction

In this chapter we discuss the methods available to
solve the mild slope equation, and its hyperbolic and
parabolic derivations. We first consider the
techniques which have been applied to the case without
currents, and then to the situation with currents
included in the formulation. It should be noted that
fewer mathematical representations of the case with
currents have been presented, and therefore only a
limited number of numerical solution techniques have
been explored.

3.2 No current action solution methods

3.2.1 Elliptic form

In work by Berkhoff (Ref 3) and others (see
Refs 16 and 17) the elliptic form of the mild slope
equation is solved numerically for several test
problems. One of these is the solution for waves
propagating over a submerged circular shaped shoal.
The solution uses a hybrid finite element approach in
which the 'inner region' problem is solved by the

finite element method. This is using a standard
technique, based on a variational principle, involving
the minimisation of a functional corresponding to the
elliptic mild slope equation. On the assumption of
deep or constant depth water the 'outer region' and
boundary values between the two areas are calculated
using a source distribution method for solving
Helmholtz equation, such that the Sommerfeld radiation
condition is satisfied.

This semi analytic matching process is also adopted by
Chen and Mei (Ref 18) for the solution of the long
wave linear shallow water equations. Representing the
boundary by a series expansion of a Hankel function,
which satisfies the radiation condition, this hybrid
finite element method solves for diffraction only.
Later Houstin (Ref 19) applied this solution technique
to the mild slope equation for refraction diffraction
type problems. Tsay and Lui (Ref 20) applied the same
matching process in the solution of the mild slope
equation in the study of wave scatter by islands and
calculation of wave force on offshore structures.

Bettess and Zienkiewicz (Ref 21) approached the
problem of solving the mild slope equation slightly
differently. The whole region was discretized using a
combination of isoparametric finite and infinite
elements over the inner region, and the outer
(constant depth) region respectively. Infinite
element are special elements which extend towards
infinity. Their associated shape functions are made
to satisfy the Sommerfeld radiation condition, and so
the whole region can be represented by one equation,
and solved by one solution method.

Greater understanding of the mild slope equation in
terms of conservation of complex energy resulted in
the reformulation by Behrendt (Ref 22) of the

functionals corresponding to those used in the variational principles in Reference 3 and Reference 21. Accurate treatment of boundary conditions and dissipation due to bed friction were also considered.

Williams et al (Ref 23) developed a implicit finite difference method to solve the mild slope equations. The model was then verified by the comparison of results with experimental data for the problem of waves passing over a circular shoal. No information is presented on the time taken to run the model.

In most cases the time taken to solve an elliptic equation using either finite difference or finite element techniques was found to be long. This was one of the reasons which prompted Saville (Ref 24) to attempt to apply the multigrid acceleration technique to finding a numerical solution of the mid slope equation. Multigrid methods are specifically designed to solve elliptic problems, and accelerate convergence by transferring the problem onto a coarser grid where high frequency errors are reduced. By repeating this process on several meshes a considerable reduction in the time taken to find a solution over more conventional methods is achieved. The method is widely used in aerodynamic applications, but its full potential is still to be realised in water wave problems.

3.2.2 Hyperbolic form

Parallel to the development of numerical solutions to the mild slope equation, various finite difference methods were applied to the hyperbolic form of the equation by Ito and Tanimoto (Ref 7). Application of an explicit scheme which includes fully and partially (Ref 8). This method calculates the integrated

component velocities at a time $\Delta t/2$ ahead of the corresponding values of η, the surface elevation. Time stepping over several wave periods is necessary to achieve steady results. This staggered mesh scheme and mid-time method is also adopted by Watanabe and Maruyam (Ref 9). Their results are in good agreement for the sloping bed problem described in Ito and Tanimoto Ref 7).

Madsen and Larsen (Ref 25) make a thorough examination of finite difference schemes suitable for the solution of their reduced set of equations based on those in Reference 7. They conclude that a forward centred difference scheme and a time varying time step leads to a stable ADI algorithm, which was found to be faster than existing solution techniques.

3.2.3 Parabolic

Finally we are concerned with the numerical solution parabolic representation given by Radder in Reference 10. Two alternative finite difference solution methods are given. The first, which deals directly with the parabolic equation, uses a Crank-Nicholson finite difference scheme given by Richtmyer and Morton (Ref 26).

By changing the description of motion, by a change of variable, to one in terms of amplitude and phase, an alternative implicit scheme is defined. This requires the addition of a dissipative term to introduce numerical damping to ensure stability. Resulting from the application of either the direct or indirect method a system of simultaneous linear equations requires solving.

The implicit Crank Nicholson scheme is also applied by Dodd (Ref 27), in which it is concluded that the

method's popularity is due to the simplistic two-level six point stencil on which it is solved. This results in low computational costs, and unconditional stability. Dodd also presents an alternative scheme based on the Douglas' equations, but despite greater accuracy it requires boundary conditions of the same order of accuracy and can only be used for parabolic models in cases of constant water depth.

3.3 With current action solution methods

3.3.1 Elliptic form

Advances in computer power, optimisation and vectorisation of computer codes has increased the feasibility of solving elliptic type problems using unsophisticated schemes. In Kostense et al (Ref 13) the development of a numerical procedure, based on a finite element method previously described in Kostense et al (Ref 28) for numerical solution of wave propagation problems in and around harbours of variable depth and current is given. A standard finite element approach using linear triangular elements is applied to equation (19). The effects of currents are introduced by means of an iterative procedure giving a approximation of the wave number vector $\underline{k}$ until certain convergence criterion are satisfied. However, to achieve reasonable solution times the models were run on a Cray supercomputer. This type of computing power is not generally available and, as a result alternatives to the full elliptic model have been the more popular means of solving the problem to date.

3.3.2 Hyperbolic form

The numerical procedures developed for the solution of
the equations presented by Dong (Ref 14) and Ohnaka et
al (Ref 15) are based on similar explicit time
stepping finite difference schemes to those previously
used to solve hyperbolic representations with no
current action.

The hyperbolic approximation given in Reference 14 to
the elliptic extended mild slope equation (Ref 12) is
discretised on a typical rectangular grid mesh
resulting in a scheme almost identical to that given
in Reference 7 and Reference 8 with the addition of
extra terms due to the effect of the current.

Discussion in (Ref 15) of the use of explicit and
implicit central, up wind, and ADE difference schemes
concludes that ADE schemes are best suited since
extensions to two dimensional wavefields are
straightforward for explicit schemes, and no numerical
diffusive terms are necessary. Along boundaries,
where the ADE scheme is unsuitable and backward
difference schemes have a tendency to produce
unrealistic reflections, the calculation region is
extended by half an interval. Thus the boundary
condition there is given in terms of surface elevation
using the method of characteristics instead of the
flow rate Q.

3.3.3 Parabolic

In both Booij (Ref 11) and Kirby (Ref 12) the
application of a finite difference method to their
parabolisations of the extended mild slope equation
are described. The preference shown in Reference 11
lies with implicit schemes because of the inherent
stability and because the resulting set of equations

can be solved extremely efficiently. Discretisation
in the x and y directions are considered independently
with presentation of two implicit schemes, the widely
used Crank Nicholson scheme and the less known Stone
and Brian scheme. The model CREDIZ (Ref 16) is based
on the parabolic approximation in Reference 11, but a
slightly more implicit scheme, that of Patankar and
Baliga, which produces the accuracy of Crank Nicholson
for small time steps and the physical realism of the
fully implicit method for larger time steps is
applied. The scheme possesses the stability of both
methods and so reduces unrealistic oscillating
solutions when the forward step Δx is large.

3.4 Discussion

From the foregoing description of available solution
methods it is clear that those for the parabolic
approximation are the most efficient. However, the
loss in information due to its assumptions made in
deriving the parabolic form mean that the parabolic
representation is only suitable for problems in which
there is unlikely to be a reflected wave field.

The solution of hyperbolic representations are slow to
run, and tests have shown that for certain cases the
solution may not converge. This is not necessarily
due to the solution method, but more likely to be
caused by the introduction of the time dependence for
what is essentially a time independent problem.

The best representation of the physical problem, the
elliptic mild slope equation, requires the greatest
computation effort for its numerical solution. This
suggests that the full elliptic equations should be
retained, and the numerical methods optimised to give
a more efficient means of solving the problem. This
could be achieved either through use of parallel

computing techniques, and/or the recently developed
multigrid methods.

4   VERIFICATION OF
REPRESENTATION

4.1 Available data

A number of almost standard example cases exist to
test the numerical models ability to reproduce certain
wave climate features without current effects. These
include the comparison with experimental observations
of refraction of waves due to a circular or elliptic
shaped shoal, and the diffraction of waves due to a
semi infinite breakwater (Refs 2,7,10,20,21,23). Some
field data examples provide information on the
combination of refraction and diffraction for real
bathymetries and offer a possibly alternative to
experimental and analytic validation.

However, no analytic and little experimental or field
data exist for the verification of models with current
fields. At best only qualitative analysis has been
possible. This is likely to continue until further
experimental work has been carried out. Martin et al
(Ref 29) present a verification procedure for wave
prediction models based on laboratory data,
hypothetical bathymetries and field data, but
inclusion of a current field is limited to a
sensitivity analysis of numerical models for the
hypothetical bathymetry.

Similarly, a series of measured wave heights for
different conditions in an area off the Haringvliet
have been presented in Reference 16 in the
verification of the numerical model CREDIZ and
recently by Holthuijsen et al (Ref 30) and Booij et al
(Ref 31) in the verification of the Hindcast Model
HISWA. Only limited verification of the effects of

currents were made in the form of a sensitivity
analysis with the conclusion given by Holthuijsen that
further field work is necessary.

Such is the difficulty in measuring waves an a moving
field most of the experimental type research is
restricted to wave profiles in the z-x plane, see
Simons (Ref 32). Since we are interested in waves
propagation in the horizontal (x-y) plane the shoaling
effects modelled in the x-z plane can only give an
indication of the change in wave characteristics as a
result of a current field.

Three reports which present some measured field data
which typify those available are described below.
Vincent (Ref 33) presents a set of observed wave
height and current speed at two adjacent locations in
the Southern North Sea. Data from a 16 day period was
analysed statistically giving a table of the
difference in mean wave height for counter-current and
co-current conditions for a number of wind speeds and
an energy spectra from the time series of maximum wave
heights.

Lambrakos (Ref 34) presents velocity frequency spectra
for waves in the Strait of Juan de Fuca, between
Washington State and Vancouver Island. Observations
made over 14 days indicate the tidal currents have a
strong influence on the wave climate in that region.
The area under these spectra decrease or increase to
the magnitude of current for co-currents and
counter-currents respectively, making this a suitable
qualatitive test for any region.

A similar case study was reported by Gonzalez
(Ref 35). The Columbia River Entrance on the
Washington-Oregon coast where the effects of currents
are known to produce considerable wave heights were

observed for a period of 5 days or ten complete tidal
cycles.

4.2 Discussion

There are available only a small number of limited
data sets for the verification of a numerical model
including waves and currents. For idealised
bathymetries and current fields it will be possible to
compare any model results with those from analytical
solutions. However, for a real bathymetry such a full
verification will not be possible as existing field
data is inadequate for this purpose. In this
situation it may only be possible to make a
qualitative assessment of the behaviour of a wave
current model for realistic bathymetries.

5 CONCLUSIONS AND RECOMMENDATIONS

1.   An assessment has been made of the mathematical
     representations which are available to model the
     effects of currents in wave disturbance models.
     It was concluded that for situations where seabed
     diffraction is not an important physical process
     then ray methods will give a reasonable
     representation. Where this phenomena is
     significant, for example in harbour approaches
     with dredged channels, then the best
     representation of the physical processes is
     provided by the mild slope equation. This is the
     case for situations both with and without
     currents.

2.   It is relatively straightforward to extend
     existing ray models of wave disturbance to
     include current effects. To develop a model
     based on the mild slope equation which includes
     currents is less easy, but will provide a

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comprehensive representation of the important physical processes.

3. Whilst it is presently computationally more efficient to solve either the hyperbolic or parabolic form of the mild slope equation, neither of these provide as a full representation of the physical processes as the elliptic form.

4. It is therefore recommended, as a first stage, that existing ray models should be extended to include current effects. This will provide a good first estimate of wave conditions, where currents are significant, suitable for many engineering purposes.

   At the same time that a mathematical model should also be developed based on the elliptic form of the mild slope equation. This will give a more complete representation of wave and current effects within and in the approaches to a harbour, which can be used in a wider range of situations than the ray models. This model should solve the governing equations using a finite difference approach. Attempts should be made to overcome problems with excessively long run-times by investigating the use of parallel algorithms or multigrid techniques.

5. The models to be developed can be verified for idealised situations against available analytical and experimental results. For realistic bathymetries there are very few field data sets available. Opportunities should be sought to allow a wave-current model for harbour studies to be more completely verified against field data.

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