Mathematical Modelling of River-Aquifer Interactions

M Nawalany

Report SR 349
March 1993

HR Wallingford

Registered Office: **HR Wallingford Ltd.** Howbery Park, Wallingford, Oxfordshire, OX10 8BA, UK
Telephone: 0491 35381 International + 44 491 35381 Telex: 848552 HRSWAL G.
Facsimile: 0491 32233 International + 44 491 32233 Registered in England No. 1622174

Contract

This report describes work carried out at HR Wallingford under the collaboration agreement between the Institute of Environmental Engineering of the Warsaw University of Technology, Poland and HR Wallingford. The work was performed by Professor Nawalany as part of a research programme on river-groundwater interaction carried out by the groundwater modelling group of Dr Charles Reeve.

Prepared by R Bettess (name) River Group (job title)

Approved by .... Senior Scientist

Date 2/7/93

Summary

Mathematical Modelling of the River-Aquifer Interaction

M Nawalany

Report SR 349
March 1993

Modelling the river-aquifer interaction is always a challenge for both practitioners and theoreticians of hydrogeology. The aim of this report is to quantify the difference between the horizontal two-dimensional flow model and the full three-dimensional model of groundwater flow in case of the river-aquifer exchange of water. The third type boundary condition is commonly assumed for the boundary between the river and the aquifer. For the two models an analytical solution of the groundwater flow equations have been found. A comparison of the total fluxes transmitted through the aquifer calculated from the two models shows that the **two models are not equivalent**. For different sets of hydraulic and geometry parameters of the river-aquifer system, the ratio between the exact 3D-flow and the 2-D horizontal approximation of flow may be considerably less than one (in some cases the ratio drops to 0.6). No simple relationship has been found which could help in assessing the ratio for a given set of parameters. The general conclusion from the research is that to model river-aquifer interactions a full three-dimensional model of groundwater flow needs to be used to calculate the correct water flow in the aquifer. The result indicates a need for further theoretical investigations of the river-aquifer interaction phenomenon to include the extension to the case of unconfined groundwater flow and for more elaborate geometries of the river cross-sections.

Notation

c	total resistivity of the river sediments, (T)
D_a	thickness of the aquifer, (L)
D_r	depth of the river, (L)
H_r	half-width of the river, (L)
k_a	hydraulic conductivity of the aquifer, (LT^{-1})
L_a	half-length of the aquifer, (L)
p	penetration of the river, $p=D_r/D_a$, (-)
q	specific discharge, (LT^{-1})
$\hat{q}$	approximation of q obtained from the 2-D model, (LT^{-1})
q_s	specific discharge through the bed of the river (i.e. seepage intensity), (LT^{-1})
Q	exact outflow from the aquifer, (L^3T^{-1})
Q_H	horizontal flow approximation of the outflow, (L^3T^{-1})
Q_{bank}	flow rate through the river bank, (L^3T^{-1})
Q_{bed}	flow rate through the river bed, (L^3T^{-1})
$\hat{Q}_H$	horizontal flow for the simplified model, (L^3T^{-1})
$\tilde{Q}(\tilde{Q}_H)$	asymptotic flow rate, (L^3T^{-1})
T_a	transmissivity of the aquifer, (L^2T^{-1})
T_r	transmissivity of the aquifer below the river bed, (L^2T^{-1})
β	auxiliary variable, (-)
λ	leakage factor, (L^2)
$\tilde{\lambda}$	leakage factor under the river, (L^2)
χ^2	modified leakage factor, (L)
ϕ	piezometric head, (L)
ϕ^o	piezometric head specified at the end of the aquifer, (L)
ϕ_r	water table position in the river, (L)
ϕ^*	auxiliary value for ϕ, (L)
$\tilde{\phi}$	modified piezometric head, ($\tilde{\phi} = \phi_r - \phi$), (L)

Contents

Page

Contents *continued*

1 Introduction

The representation of river-aquifer interactions (r-a-i) always poses problems when modelled as a part of some larger hydrological systems. Especially when incorporated into the two-dimensional horizontal flow models of groundwater systems, the (r-a-i) may not be mimicked accurately thus resulting in an unacceptable inaccuracy of the global mass balance. It is the intention of this report to check whether (r-a-i) can be represented within a framework of horizontal flow models. For the simple case of a river that recharges the adjacent aquifer two analytical models - the two-dimensional horizontal and the three-dimensional one - are derived. The models are compared with each other in terms of the total seepage within a series of computer experiments. Also the asymptotic behaviour of the two models is analyzed showing conformity of the models with the physical background of the river-aquifer interaction. From the calculations important recommendations on the experimental and modelling aspects of the (r-a-i) are drawn.

2 Assumptions

The following figure (Figure 2.1) illustrates the case that is analyzed through the report.

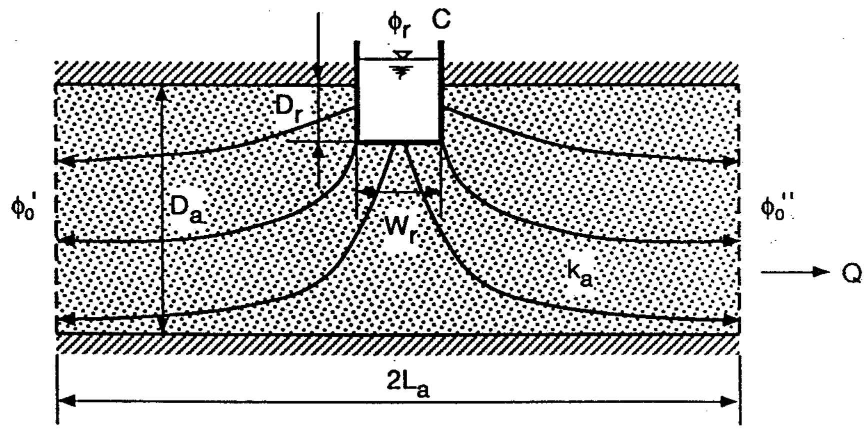

Figure 2.1 River-aquifer system.

Assumptions

Throughout this report a river bed is assumed to have a rectangular cross-section. It is located in the middle of the rectangular, homogeneous and confined aquifer and it only partially penetrates the aquifer. Also a constant water table position is assumed in the river. At both ends of the aquifer constant piezometric heads - ϕ'_o and ϕ''_o - are specified. They are assumed to be equal to each other and less than the piezometric head in the river, i.e. $\phi'_o = \phi''_o = \phi_o < \phi_r$. This implies that the interaction between the river and the aquifer is symmetric in space. Physically the interaction is a seepage through the sediments which have accumulated on at the river bed and banks. The seepage is proportional to the difference between the piezometric heads in the river and the aquifer and reciprocal to the flow resistivity through the sediments - c. The resistivity c is also assumed to be homogeneous. Because of the geometric symmetry of the case, only one half of the (r-a-i) system needs to be considered - Figure 2.2.

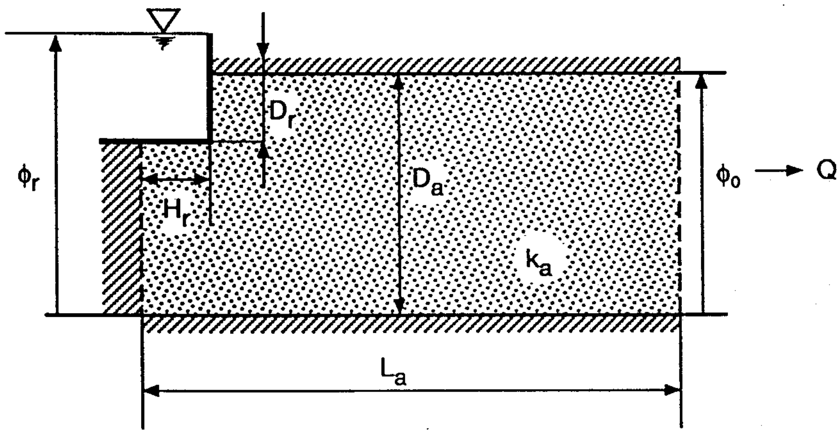

Figure 2.2 River-aquifer system (simplified).

This figure is repeated throughout the report showing distinct features of the models being considered.

3 *Simple two-dimensional model of the river-aquifer interaction*

For the special case of a shallow river - see Figure 3.1 - with its dimensions negligible when compared with the dimensions of the adjacent aquifer, a water continuity requirement is sufficient to derive a formula for the aquifer outflow $\hat{Q}_H$.

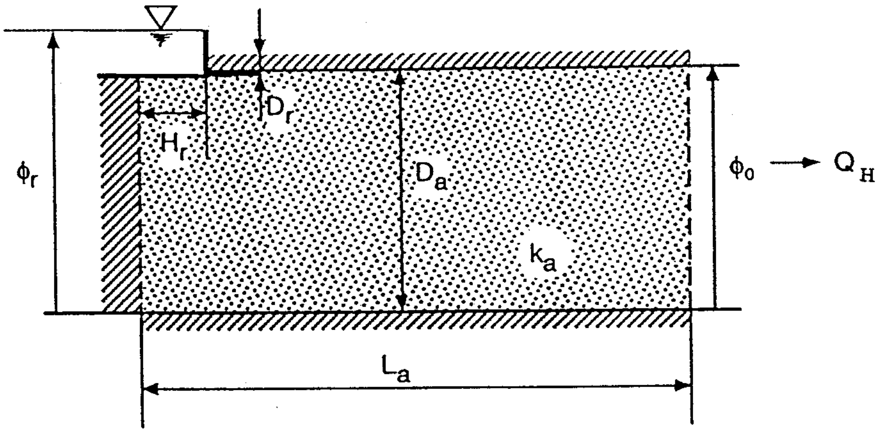

Figure 3.1 Shallow river - aquifer interaction (2D-simple).

Assumptions:

i) $D_r << H_r$
ii) $D_r << D_a$
iii) $H_r << L_a$
iv) ϕ^* = const. below the bed of the river.

The continuity requirement can be formulated as follows:

$$\hat{Q}_H = Q_{bed} \tag{3.1}$$

$$D_a \cdot k_a \cdot \frac{\phi^* - \phi_o}{L_a} = \frac{\phi_r - \phi^*}{c} \cdot H_r \tag{3.2}$$

From (3.2) the unknown ϕ^* can be calculated from the equation

$$\phi^* = \frac{\phi_o \lambda^2 + \phi_r L_a H_r}{\lambda^2 + L_a H_r} \tag{3.3}$$

where,

$$\lambda^2 = T_a \cdot c$$
$$T_a = D_a \cdot k_a .$$

After substituting (3.2) into (3.3) one gets

$$\hat{Q}_H = \frac{T_a}{L_a}(\phi_r - \phi_o)\frac{L_a H_r}{\lambda^2 + L_a H_r} \tag{3.4}$$

This is the required formula for the shallow river-aquifer interaction.

4 Complete two-dimensional model of the river-aquifer interaction

For this case we relax all the assumptions made in the previous chapter. Now the river can have arbitrary dimensions and ϕ^* (aquifer's piezometric head below the river's bottom) may be a function of x. Still, a piezometric head in the aquifer is considered depth-averaged. Also Q_{bank} contributes to the global outflow Q_H - see Figure 4.1.

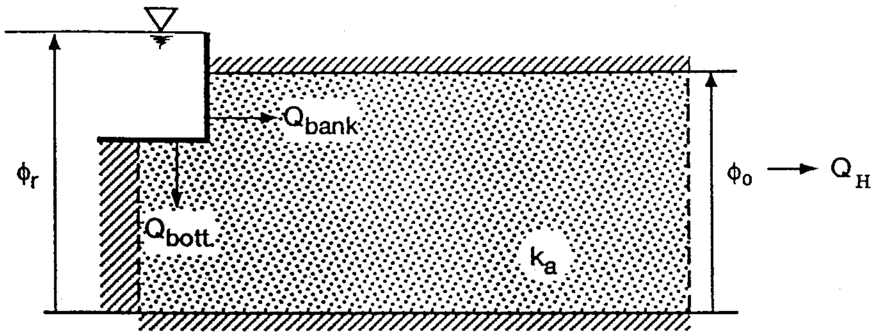

Figure 4.1 River-aquifer interaction (2D-complete).

4.1 Analytical solution for the flow equation

If ϕ^* is a value of piezometric head at $x = H_r$ the total outflow Q_H can be calculated from the following formula:

$$Q_H = T_a \cdot \frac{\phi^* - \phi_o}{L_a - H_r} \tag{4.1}$$

where $T_a = D_a \cdot k_a$.

An analytical horizontal flow model is derived by considering the 2D-flow equation with the source term (seepage flux) q_s for $x \in [0,H_r]$ - see also Figure 4.2:

$$q_s(x) = \frac{\phi_r - \phi}{c} \tag{4.2}$$

The steady state flow equation for this region is therefore

$$T_r \frac{\partial\phi}{\partial x^2} + \frac{\phi_r - \phi}{c} = 0 \tag{4.3}$$

where $T_r = (D_a - D_r) \cdot k_a$. (4.4)

The boundary conditions for the region are as follows:

$$\begin{cases} \frac{\partial\phi}{\partial x}\Big|_{x=0} = 0 \\ \phi\,\Big|_{x=H_r} = \phi^*. \end{cases} \tag{4.5}$$

By introducing the notation

$$\tilde{\phi} = \phi_r - \phi \tag{4.6a}$$

and

$$\tilde{\lambda}^2 = T_r \cdot c \ , \tag{4.6b}$$

equation (4.3) can be rewritten as:

$$-\frac{\partial^2\tilde{\phi}}{\partial x^2} + \frac{\tilde{\phi}}{\tilde{\lambda}^2} = 0 \tag{4.7}$$

with the boundary conditions (b.c.-s):

$$\begin{cases} \frac{\partial\tilde{\phi}}{\partial x}\Big|_{x=0} = 0 \quad \text{and} \\ \tilde{\phi}\,\Big|_{x=H_r} = \phi_r - \phi^*. \end{cases} \tag{4.8}$$

The general solution to equation (4.7) has the form:

$$\tilde{\phi}(x) = Ae^{x/\tilde{\lambda}} + Be^{-x/\tilde{\lambda}} \tag{4.9}$$

The constants A and B can be calculated from the boundary conditions as follows:

$$\text{(i)} \quad 0 = \frac{\partial\tilde{\phi}}{\partial x}\Big|_{x=0} = \left(\frac{A}{\tilde{\lambda}}e^{x/\tilde{\lambda}} - \frac{B}{\tilde{\lambda}}e^{-x/\tilde{\lambda}}\right)\Big|_{x=0} = (A-B)/\tilde{\lambda}$$

$$\Rightarrow \quad A=B \quad \Rightarrow \quad \tilde{\phi}(x) = A(e^{x/\tilde{\lambda}} + e^{-x/\tilde{\lambda}}).$$

Hence the general solution has the form

$$\tilde{\phi}(x) = \tilde{A}\cosh(x/\tilde{\lambda}). \qquad (4.10)$$

$$\text{(ii)} \quad \phi_r - \phi^* = \tilde{\phi}(H_r) = \tilde{A}\cosh(H_r/\tilde{\lambda})$$

$$\Rightarrow \quad \tilde{A} = \frac{\phi_r - \phi^*}{\cosh(H_r/\tilde{\lambda})}.$$

Finally, we get

$$\tilde{\phi}(x) = (\phi_r - \phi^*)\,\frac{\cosh(x/\tilde{\lambda})}{\cosh(H_r/\tilde{\lambda})}. \qquad (4.11)$$

After returning to the original piezometric head $\phi(x)$ we have

$$\phi(x) = \phi_r - (\phi_r - \phi^*)\,\frac{\cosh(x/\tilde{\lambda})}{\cosh(H_r/\tilde{\lambda})}. \qquad (4.12)$$

It should be noted here that the total flow increases instantly at the point $x=H_r$ because of the additional bank seepage. This causes an abrupt change in the slope of ϕ when passing from $x=H^-_r$ to $x=H_r^+$ - see Figure 4.2.

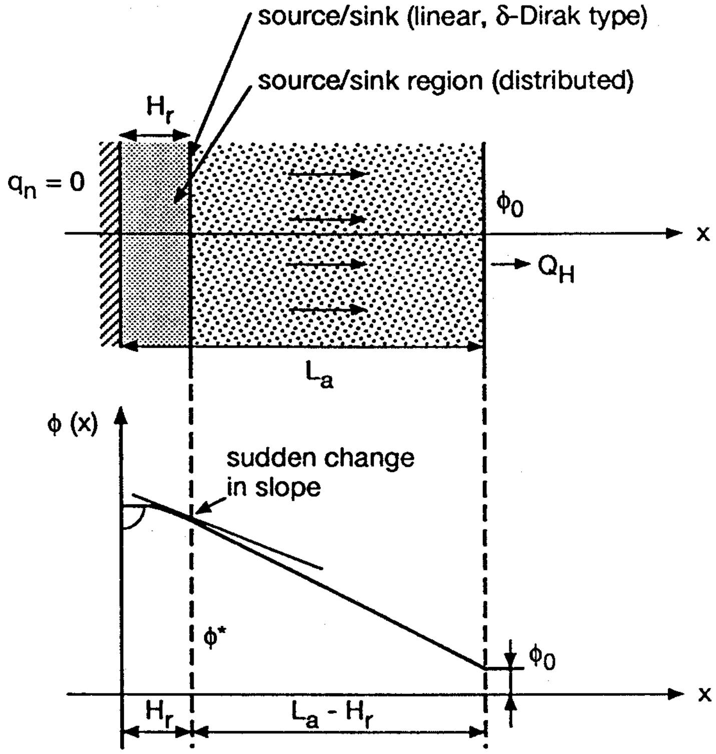

Fig.4.2. Top view of an aquifer and piezometric head for the 2D-horizontal flow model.

The unknown ϕ^* can be calculated from the total mass balance equation

$$Q_H = Q_{bed} + Q_{bank} \tag{4.13}$$

where,

$$Q_{bed} = (D_a - D_r) \cdot k_a \cdot \left(-\frac{\partial \phi}{\partial x} \Big|_{x=H_r^-} \right) = T_r \cdot (\phi_r - \phi^*) \, \mathrm{tgh}(H_r/\tilde{\lambda})/\tilde{\lambda} \tag{4.14}$$

$$Q_{bank} = D_r \cdot \frac{(\phi_r - \phi^*)}{c} \tag{4.15}$$

$$Q_H = T_a \frac{\phi^* - \phi_o}{L_a - H_r} \quad \text{- formula (4.1).}$$

After substituting (4.1),(4.14) and (4.15) into (4.13) we obtain

$$T_a \frac{\phi^* - \phi_o}{L_a - H_r} = T_r (\phi_r - \phi^*) \, \mathrm{tgh}(H_r/\tilde{\lambda})/\tilde{\lambda} + D_r \frac{(\phi_r - \phi^*)}{c}$$

from which

$$\phi^* = \frac{\phi_o + \phi_r \psi}{1 + \psi} \tag{4.16}$$

where

$$\psi = \frac{L_a - H_r}{T_a}\left\{\frac{T_r}{\tilde{\lambda}}\,\mathrm{tgh}(H_r/\tilde{\lambda}) + \frac{D_r}{c}\right\} \tag{4.17}$$

$$T_r = (D_a - D_r)k_a = \left(1 - \frac{D_r}{D_a}\right)D_a k_a = (1-p)T_a \tag{4.18}$$

$$p = \frac{D_r}{D_a} \quad \text{- penetration} \tag{4.19}$$

$$\tilde{\lambda}^2 = T_r c = (1-p)T_a c = (1-p)\lambda^2 \quad \text{- corrected leakage factor} \tag{4.20}$$

$$\lambda^2 = T_a c \quad \text{- normal leakage factor.}$$

Formula (4.1) together with formulae (4.16)-(4.18) define the required solution to the 2D-horizontal flow equation that include the river-aquifer interaction.

Remark 1. Since $\psi > 0$ therefore

$$\phi_o < \phi^* < \phi_r \tag{4.21}$$

Remark 2. Formula (4.14) can also be obtained by integrating the seepage along the bed of the river. Indeed,

$$Q_{bed} = \int_0^{H_r} \frac{\phi_r - \phi(x)}{c}\,dx = \int_0^{H_r} \frac{\phi_r - \phi^*}{c}\,\frac{\cosh(x/\tilde{\lambda})}{\cosh(H_r/\tilde{\lambda})}\,dx$$

$$= \frac{\phi_r - \phi^*}{c}\cdot\frac{1}{\cosh(H_r/\tilde{\lambda})}\cdot\tilde{\lambda}\sinh(H_r/\tilde{\lambda}) = \tilde{\lambda}^2\cdot\frac{\phi_r - \phi^*}{\tilde{\lambda}c}\cdot\mathrm{tgh}(H_r/\tilde{\lambda})$$

$$= T_r\cdot c\cdot\frac{\phi_r - \phi^*}{\tilde{\lambda}\cdot c}\,\mathrm{tgh}(H_r/\tilde{\lambda}) = \text{formula (4.14).}$$

Remark 3. In the particular case of the fully penetrating river, i.e. when $p \to 1$, we have

$$T_r \to 0 \text{ and } \lambda \to 0 \Rightarrow \tilde{\lambda} \to 0 \Rightarrow \mathrm{tgh}(H_r/\tilde{\lambda}) \to 1$$

$$\Rightarrow Q_{bed} \rightarrow \frac{(1-p)T_a}{\sqrt{1-p}\,\sqrt{T_a c}} = \frac{\sqrt{1-p}\,\sqrt{T_a}}{\sqrt{c}} \rightarrow 0$$

Hence

$$p \rightarrow 1 \;\Rightarrow\; Q_{bed} \rightarrow 0.$$

Remark 4. It can be proven that the total seepage ($=Q_H$) calculated from the 2D-model is always larger than the exact Q. This however can be deduced even without deriving the formulae for 3D-model. Indeed, when considering the vertical distribution of piezometric head below the river bed one can observe that since $\phi(x,z)$ is an increasing function of z (or decreasing function of depth d -see Figure 4.3) the value of ϕ just below the bed (i.e. $\phi(x,D_a\text{-}D_r)$) is always larger than the vertical average of ϕ used in the 2D-model. When the vertical seepage flux for the two models is calculated from the same formula (4.2) using different values of piezometric head two different values for q_s are obtained:

2Dh: $$\hat{q}_s = \frac{\phi_r - \tilde{\phi}(x)}{c}$$

3D: $$q_s = \frac{\phi_r - \phi(x,D_a - D_r)}{c}$$

Since $\tilde{\phi}(x) < \phi(x,D_a - D_r)$ } it follows that $|\,\hat{q}_s\,| > |\,q_s\,|$.

Obviously, when a river is draining an aquifer we may repeat this reasoning:

2D: $$\hat{q}_s = \frac{\tilde{\phi}(x) - \phi_r}{c}$$

3D: $$q_s = \frac{\phi(x,D_a - D_r) - \phi_r}{c}$$

Since $\tilde{\phi}(x) > \phi(x,D_a - D_r)$ hence again $|\hat{q}_s| > |q_s|$.

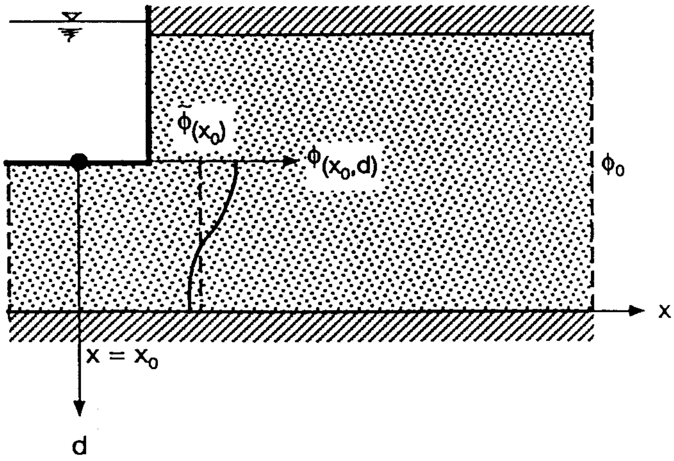

Figure 4.3 Piezometric head under a river bed.

Concluding, we may state that in any case 2D-horizontal models always lead to an overestimate of the river-groundwater interaction.

4.2 Asymptotic behaviour of the solution

From the solution (4.1), the asymptotic behaviour of the horizontal flow Q_H can be deduced for a number of special cases.

Case one. River is in good contact with the aquifer i.e. $c \rightarrow 0$

$$c \rightarrow 0 \overset{(4.17)}{\Rightarrow} \psi \rightarrow \infty \overset{(4.16)}{\Rightarrow} \phi^* \rightarrow \phi_r$$

$$\Rightarrow \tilde{Q}_H = T_a \cdot \frac{\phi_r - \phi_o}{L_a - H_r} \qquad (4.22)$$

Case two: River is isolated from the aquifer i.e. $c \rightarrow \infty$

$$c \rightarrow \infty \Rightarrow \psi \rightarrow 0 \Rightarrow \phi^* \rightarrow \phi_o$$

$$\Rightarrow \tilde{Q}_H = 0 \qquad (4.23)$$

Case three: The geometry of the river is negligible when compared with the dimensions of the aquifer i.e. $H_r/L_a \rightarrow D_r/D_a \rightarrow 0$ (but $c \neq 0$). As a consequence of $D_r/D_a = p \rightarrow 0$, $\lambda \rightarrow \lambda$, $T_r/T_a \rightarrow$ and

$$\tilde{\psi} \doteq \frac{L_a(1-H_r/L_a)}{T_a}\left\{\frac{T_rH_r}{\tilde{\lambda}^2}\underbrace{\frac{\mathrm{tgh}\left[\frac{H_r/L_a}{\tilde{\lambda}/L_a}\right]}{\left(\frac{H_r/L_a}{\tilde{\lambda}/L_a}\right)}}_{\to 1} + \underbrace{\frac{p\cdot D_a}{c}}_{\to 0}\right\} = \frac{L_aH_r}{\lambda^2}$$

From this and from (4.16) we get

$$\tilde{\phi}^* \doteq \frac{\phi_o + \phi_r \cdot L_aH_r/\lambda^2}{1 + L_aH_r/\lambda^2} \tag{4.24}$$

After substituting this in formula (4.1) we obtain

$$\tilde{Q}_H \doteq \frac{T_a}{L_a}\left[\frac{\phi_o + \phi_r L_aH_r/\lambda^2 - \phi_o - \phi_o L_aH_r/\lambda^2}{1 + L_aH_r/\lambda^2}\right]$$

from which we finally get:

$$\tilde{Q}_H \doteq T_a\frac{\phi_r - \phi_o}{L_a}\cdot\frac{L_aH_r}{\lambda^2 + L_aH_r} \tag{4.25}$$

This is exactly the formula (3.4) for the simplified model of 2D-horizontal flow.

<u>Case four:</u> The river is fully penetrating, i.e. $D_r \to D_a$

Then

$$\psi\big|_{D_a=D_r} = \frac{L_a-H_r}{T_a}\left\{0 + \frac{D_a}{c}\right\} = \frac{(L_a-H_r)D_a}{k_a\cdot D_a\cdot c} = \frac{L_a-H_r}{\chi^2}$$

where $\chi^2 = k_a \cdot c$.

From formulae (4.16) and (4.1) we obtain

$$\phi^* = \frac{\phi_o + \phi_r\frac{(L_a-H_r)}{\chi^2}}{1 + \frac{L_a-H_r}{\chi^2}} = \frac{\phi_o\chi^2 + \phi_r(L_a-H_r)}{\chi^2 + (L_a-H_r)}$$

$$\tilde{Q}_H = \frac{T_a}{L_a - H_r}\left[\frac{\phi_o\chi^2 + \phi_r(L_a - H_r) - \phi_o\chi^2 - \phi_o(L_a - H_r)}{\chi^2 + (L_a - H_r)}\right]$$

Finally we have

$$\tilde{Q}_H = T_a \frac{\phi_r - \phi_o}{\chi^2 + (L_a - H_r)} \tag{4.26}$$

This is exactly the same as the solution to the one-dimensional flow equation if the third-type boundary condition is set for the left boundary and the first-type boundary conditions is set for the right boundary.

5 Three-dimensional model of the river-aquifer interaction

In order to solve the groundwater flow problem in three dimensions we subdivide a flow domain Ω into two parts - I and II - see Figure 5.1

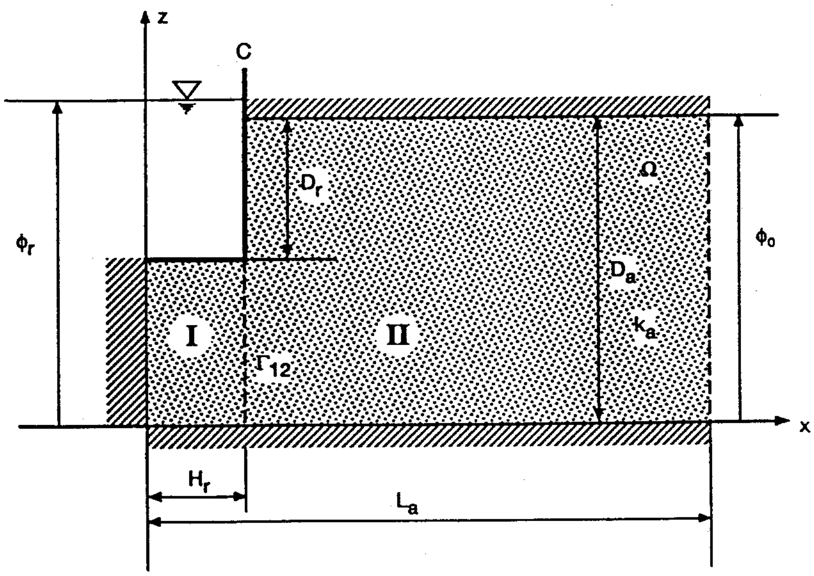

Figure 5.1 Subdivision of the 3D-flow domain.

and solve two separate flow problems for the two parts of Ω. After that we match the solutions along the common boundary Γ_{12} in terms of piezometric heads and normal fluxes, i.e.:

$$\phi^{I}(x,z) = \phi^{II}(x,z) \tag{5.1}$$

and

$$\frac{\partial\phi^{I}(x,z)}{\partial x} = \frac{\partial\phi^{II}(x,z)}{\partial x}$$

for $x=H_r$ and for all $z\varepsilon[O,D_a-D_r]$.

5.1 Analytical solution to the flow equation.

Solution for region I
Piezometric head in region I $\phi^I(x,z)$ must satisfy the flow equation:

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial z^2}=0 \quad \text{for} \quad \begin{cases} x\varepsilon[0,H_r] \\ z\varepsilon[0,D_a-D_r] \end{cases} \tag{5.2}$$

and the following boundary conditions:

$$\begin{cases} \Gamma_1: & \frac{\partial\phi}{\partial x}|_{x=0}=0 & \text{for } z\varepsilon[0,D_a-D_r] \\ \Gamma_2: & -k_a\frac{\partial\phi}{\partial z}|_{z=D_a-D_r}=\frac{\phi-\phi_r}{c}|_{z=D_a-D_r} & \text{for } x\varepsilon[0,H_r] \\ \Gamma_{12}: & \phi|_{x=H_r}=\phi^{II}|_{x=H_r} & \text{for } z\varepsilon[0,D_a-D_r] \\ \Gamma_4: & \frac{\partial\phi}{\partial z}|_{z=0}=0 & \text{for } x\varepsilon[0,H_r] \end{cases} \tag{5.2'}$$

Here and in the following the superscript "I" is omitted.
By shifting the reference level for the piezometric head we define

$$\tilde{\phi}(x,z) \hat{=} \phi(x,z)-\phi_r \tag{5.3}$$

Then equation (5.2) becomes

$$\frac{\partial^2\tilde{\phi}}{\partial x^2}+\frac{\partial^2\tilde{\phi}}{\partial z^2}=0. \tag{5.4}$$

Also, $\tilde{\phi}$ must satisfy the following boundary conditions:

$$\begin{cases} \Gamma_1: & \frac{\partial\tilde{\phi}}{\partial x}|_{x=0}=0 & \text{for } z\varepsilon[0,D_a-D_r] \\ \Gamma_2: & -\frac{\partial\tilde{\phi}}{\partial z}|_{z=D_a-D_r}=\frac{\tilde{\phi}}{\chi^2}|_{z=D_a-D_r} & \text{for } x\varepsilon[0,H_r] \\ \Gamma_{12}: & \tilde{\phi}|_{x=H_r}=\tilde{\phi}^{II}|_{x=H_r} & \text{for } z\varepsilon[0,D_a-D_r] \\ \Gamma_4: & \frac{\partial\tilde{\phi}}{\partial z}|_{z=0}=0 & \text{for } x\varepsilon[0,H_r] \end{cases} \tag{5.4'}$$

where

$$X^2=k_a.c \text{ - modified leakage factor, (m).} \tag{5.5}$$

We seek a solution in the factorized form

$$\tilde{\phi}(x,z) = X(x)\,Z(z)$$

which, after substitution to (5.4), gives

$$ZX'' + XZ'' = 0 \qquad (5.7)$$

or

$$\frac{X''(x)}{X(x)} = -\frac{Z''(z)}{Z(z)} \qquad (5.7')$$

This can only be satisfied if

$$\frac{X''}{X} = -\frac{Z''}{Z} = \lambda^2 \qquad (5.8)$$

where λ is a constant,

i.e. if

$$\begin{cases} X'' - \lambda^2 X = 0 \quad \text{and} & (5.9a) \\ Z'' + \lambda^2 Z = 0. & (5.9b) \end{cases}$$

The general solution to equation (5.9a) is given by the following formula:

$$X(x) = Ae^{\lambda x} + Be^{-\lambda x} \qquad (5.10)$$

From the boundary condition on Γ_1 we have:

$$0 = \frac{\partial\tilde{\phi}}{\partial x}\Big|_{x=0} = Z\,\frac{dX}{dx}\Big|_{x=0} = Z\,(A\lambda e^{\lambda x} - B\lambda e^{-\lambda x})\Big|_{x=0} = Z\,(A-B)\lambda \;\Rightarrow\; \underline{A = B.}$$

Hence

$$X(x) = A\cosh(\lambda x) \qquad (5.11)$$

The general solution to equation (5.9b) is given by

$$Z(z) = C\sin\lambda z + D\cos\lambda z \qquad (5.12)$$

From the boundary condition on Γ_3 we have:

$$0 = \frac{\partial Z}{\partial z}|_{z=0} = (C\lambda\cos\lambda z - B\lambda\sin\lambda z)|_{z=0} = C\lambda \Rightarrow \underline{C = 0.}$$

Consequently

$$Z(z) = D\cos\lambda z \tag{5.13}$$

Substituting (5.11) and (5.13) to (5.6) we obtain the general form of the required solution in region I:

$$\tilde{\phi}(x,z) = A\cosh(\lambda x)\cos(\lambda z) \tag{5.14}$$

The solution should satisfy the boundary condition on Γ_2, i.e. it should satisfy the chosen model of the river-aquifer interaction - see formula (5.4'):

$$-A\cosh(\lambda x)[-\lambda\sin(\lambda z)]|_{z=D_a-D_r} = \frac{A\cosh(\lambda x)\cos(\lambda z)}{\chi^2}|_{z=D_a-D_r}$$

$$\Rightarrow \frac{\sin[\lambda(D_a-D_r)]}{\cos[\lambda(D_a-D_r)]} = \frac{1}{\lambda\chi^2} \tag{5.15}$$

Formula (5.15) can be equivalently written as

$$tg\tilde{\lambda} = \frac{D_a-D_r}{\chi^2}\cdot\frac{1}{\tilde{\lambda}} \tag{5.15'}$$

where

$$\tilde{\lambda} = \lambda(D_a-D_r). \tag{5.16}$$

There is an infinite number of $\tilde{\lambda}$-s that satisfy (5.15'). This is clearly indicated in the following Figure 5.2.

Consequently, there is an infinite number of λ-s that satisfy (5.15)

$$\lambda_k = \frac{\tilde{\lambda}_k}{D_a-D_r}; \quad (k=1,2,...) \tag{5.17}$$

where $\tilde{\lambda}_k$ - denotes solutions of equation (5.15').

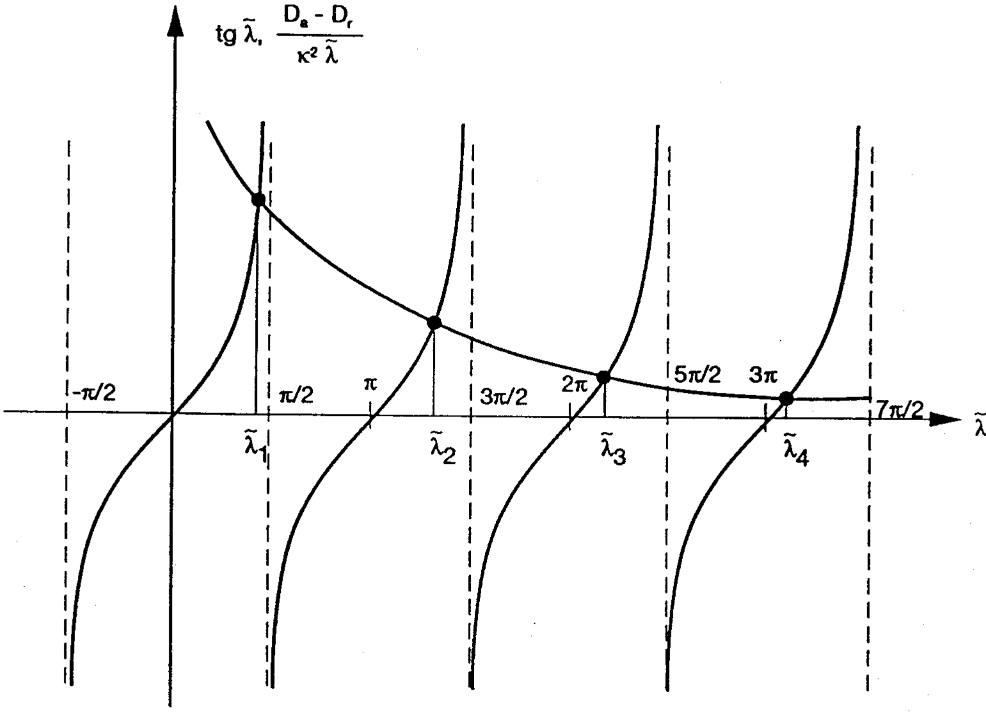

Figure 5.2 Graphical interpretation to the nonlinear equation (5.15').

Finally, the solution for region I can be expressed as an infinite series:

$$\phi(x,z) = \phi_r + \sum_{k=1}^{\infty} A_k \cosh(\lambda_k x) \cos(\lambda_k z) \quad \text{for } 0<x\leq H_r, \quad 0<z\leq D_a - D_r. \quad (5.18)$$

where A_k (k=1,2,...) are unknown constants that will be calculated from matching condition (5.1).

Remark 1: It can be easily checked that solution (5.18) does satisfy the water mass balance for the region I. Indeed,

$$Q_{in} = \int_0^{H_r} k_a \frac{\partial\tilde{\phi}}{\partial z}|_{z=D_a-D_r} dx = k_a \int_0^{H_r} \left(-\sum_{k=1}^{\infty} A_k \lambda_k \cosh(\lambda_k x) \sin\lambda_k (D_a - D_r) \right) dx =$$

$$= -k_a \sum_{k=1}^{\infty} A_k \lambda_k \sin[\lambda_k (D_a - D_r)] \int_0^{H_r} \cosh(\lambda_k x)\, dx =$$

$$= -k_a \sum_{k=1}^{\infty} A_k \sin[\lambda_k (D_a - D_r)] \cdot \sinh(\lambda_k H_r)$$

(5.19)

$$Q_{out} = \int_0^{D_a-D_r} -k_a \frac{\partial\phi}{\partial x}|_{x=H_r} dx = -k_a \int_0^{D_a-D_r} \sum_{k=1}^{\infty} A_k \lambda_k \sinh(\lambda_k H_r) \cos(\lambda_k z) dx =$$

$$= -k_a \sum_{k=1}^{\infty} A_k \lambda_k \sinh(\lambda_k H_r) \int_0^{D_a-D_r} \cos(\lambda_k z) dx = \qquad (5.20)$$

$$= -k_a \sum_{k=1}^{\infty} A_k \sin[\lambda_k (D_a - D_r)] \sinh(\lambda_k H_r)$$

Hence

$$Q_\epsilon = Q_{out}$$

Solution for region II

The piezometric head $\tilde{\phi}^{II} = \phi^{II} - \phi_r$ (abbreviated hereafter as $\tilde{\phi}$) must satisfy the flow equation:

$$\frac{\partial\tilde{\phi}}{\partial x^2} + \frac{\partial^2\tilde{\phi}}{\partial z^2} = 0 \quad \text{for} \begin{cases} x \varepsilon [H_r, L_a] \\ z \varepsilon [0, D_a] \end{cases} \qquad (5.21)$$

boundary conditions:

$$\begin{cases} \Gamma_{12}'': & \frac{\partial\tilde{\phi}}{\partial x}|_{x=H_r} = \frac{\tilde{\phi}}{\chi^2}|_{x=H_r} & \text{for } z\varepsilon[D_a-D_r,D_a] \\ \Gamma_2: & \frac{\partial\tilde{\phi}}{\partial z}|_{z=D_a} = 0 & \text{for } x\varepsilon[H_r,L_a] \\ \Gamma_3: & \tilde{\phi}|_{x=L_a} = \phi_o-\phi_r = \tilde{\phi}_o & \text{for } z\varepsilon[0,D_a] \\ \Gamma_4: & \frac{\partial\tilde{\phi}}{\partial z}|_{z=0} = 0 & \text{for } x\varepsilon[H_r,L_a] \end{cases} \tag{5.21'}$$

and the matching condition (5.1).

By factorizing the solution for $\tilde{\phi}^{II}(x,z)$ in similar way as for $\tilde{\phi}^{I}$ we obtain

$$\frac{X''}{X} = -\frac{Z''}{Z} = \mu^2. \tag{5.22}$$

As before, the general solutions for X(x) and Z(z) have the form:

$$X(x) = \tilde{A}e^{\mu(L_a-x)} + \tilde{B}e^{-\mu(L_A-x)} \tag{5.23}$$

and

$$Z(z) = \tilde{C}\sin\mu z + \tilde{D}\cos\mu z. \tag{5.24}$$

From the boundary condition on Γ_4 we get

$$0 = \frac{\partial Z(z)}{\partial z}|_{z=0} = (\tilde{C}\mu\cos\mu z - \tilde{D}\mu\sin\mu z)|_{z=0} = \tilde{C}\mu \Rightarrow \tilde{C} = 0.$$

Therefore (5.24) becomes

$$Z(z) = \tilde{D}\cos\mu z \tag{5.25}$$

From the boundary condition on Γ_2 we get

$$0 = \frac{\partial Z(z)}{\partial z}|_{z=D_a} = -\tilde{D}\mu\sin\mu D_a$$

which can be satisfied only if $\sin\mu D_a=0$ i.e. when $\mu D_a = l\pi$.

Hence

$$\mu_l = \frac{l\pi}{D_a} \;; \quad (l=1,2,...) \tag{5.26}$$

Remark 2. In (5.26) l=0 has been omitted as it only introduces a constant to the solution whereas l<0 has been omitted because the function (5.25) is even.

Combining (5.23) and (5.25) for all possible μ_l given by (5.26) we obtain the required solution for region II:

$$\tilde{\phi}(x,z) = \sum_{l=1}^{\infty} \left[\tilde{A}_l e^{\mu_l(L_a-x)} + \tilde{B}_l e^{-\mu_l(L_a-x)}\right] \cos\mu_l z \tag{5.27}$$

Without violating the boundary conditions for Γ_2 and Γ_4 we can add a linear term L(x) to the solution

$$L(x) = \alpha + \beta^* x \tag{5.28}$$

After combining (5.27) and (5.28) and assigning temporarily $\tilde{\phi}$ given by (5.27) as $\hat{\phi}(x,z)$ we get the solution for region II in its generalized form:

$$\tilde{\phi}(x,z) = \hat{\phi}(x,z) + L(x) \tag{5.29}$$

If we force L(x) to become $\tilde{\phi}_o = \phi_o - \phi_r$ for $x=L_a$ we enforce a zero boundary condition for $\hat{\phi}(x,z)$ for $x=L_a$ and all $z \in [0,D_a]$. In other words

$$L(L_a) = \tilde{\phi}_o = \alpha + \beta^* L_a$$

results in

$$L(x) = \tilde{\phi}_o - \beta^*(L_a - x) \tag{5.30}$$

and

$$0 = \hat{\phi}(L_a,z) = \sum_{l=1}^{\infty} [\tilde{A}_l e^{\mu\rho} + \tilde{B}_l e^{-\mu\rho}] \cos\mu_l z \; .$$

The latter relationship can hold only if

$$\tilde{B}_l \;=\; -\tilde{A}_l \qquad \text{for all } l=1,2,\dots \; . \tag{5.31}$$

Consequently, the solution in region II can be expressed as:

$$\tilde{\phi}(x,z) = \sum_{l=1}^{\infty} D_l \sinh[\mu_l (L_a - x)] \cos\mu_l z + \tilde{\phi}_o - \beta^* (L_a - x) \tag{5.32}$$

Remark 3: On Γ_{12}'' of region II (i.e. along a river bank) the solution (5.32) must satisfy a river-aquifer interaction condition:

$$\frac{\partial\tilde{\phi}}{\partial x}\Big|_{x=H_r} = \frac{\tilde{\phi}}{\chi^2}\Big|_{x=H_r} \qquad \text{for } z\varepsilon[D_a - D_r, D_a]$$

i.e.

$$\begin{cases} -\sum_{l=1}^{\infty} D_l \mu_l \cosh[\mu_l(L_a - H_r)] \cos\mu_l z + \beta^* = \\ = \sum_{l=1}^{\infty} \frac{D_l}{\chi^2} \sinh[\mu_l(L_a - H_r)] \cos\mu_l z + \frac{\tilde{\phi}_o - \beta^*(L_a - H_r)}{\chi^2} \end{cases} \tag{5.33}$$

Remark 4: On the common boundary Γ_{12}' between regions I and II we must match both the piezometric heads and the normal fluxes. Before doing so we can observe that it is very convenient to represent functions $\cos(\lambda_k z)$ (in (5.18)) by the orthogonal functions $\cos \mu_l z$ (l=1,2,...). Orthogonality of the family {$\cos\mu_l z$; l=1,2,} means that for

$$J_{lk} = \int_0^{D_a} \cos(\mu_l z) \cos(\mu_k z)\, dz \tag{5.34}$$

the following conditions hold:

$$J_{lk} = \begin{cases} 0 & \text{for } l \neq k \\ \dfrac{D_a}{2} & \text{for } l = k \end{cases}$$

Remark 5: Below some useful formulae are recalled that allow one to calculate the representation of $\cos(\lambda_k z)$ in terms of $\cos(\mu_i z)$:

(i) for $a \neq b$

$$\int \cos ax \cos bx \, dx = \frac{\sin(a-b)x}{2(a-b)} + \frac{\sin(a+b)x}{2(a+b)}$$

(ii) for a=b

$$\int \cos^2 ax \, dx = \frac{x}{2} + \frac{\sin 2ax}{4a}$$

Remark 6: From (R4) and (R5) we can conclude that:

$$\text{for } i=j \quad \varphi_{ii} = \int_0^{D_a-D_r} \cos^2 \mu_i z \, dz = \frac{D_a-D_r}{2} + \frac{\sin[2\mu_i(D_a-D_r)]}{4\mu_i}$$

$$\text{for } i \neq j \quad \varphi_{ij} = \int_0^{D_a-D_r} \cos(\mu_i z)\cos(\mu_j z)\, dz = \tag{5.37}$$

$$= \frac{\sin[(\mu_i-\mu_j)(D_a-D_r)]}{2(\mu_i-\mu_j)} + \frac{\sin[(\mu_i+\mu_j)(D_a-D_r)]}{2(\mu_i+\mu_j)} =$$

$$= \frac{\sin[|\mu_i-\mu_j|(D_a-D_r)]}{2|\mu_i-\mu_j|} + \frac{\sin[(\mu_i+\mu_j)(D_a-D_r)]}{2(\mu_i+\mu_j)}$$

and additionally, that for any integers k and j

$$\gamma_{kj} = \int_{0}^{D_a-D_r} \cos(\lambda_k z)\cos(\mu_j z)\,dz = \tag{5.38}$$
$$= \frac{\sin[|\lambda_k-\mu_j|(D_a-D_r)]}{2|\lambda_k-\mu_j|} + \frac{\sin[(\lambda_k+\mu_j)(D_a-D_r)]}{2(\lambda_k+\mu_j)}$$

Now the matching conditions (5.1) between regions I and II can be written as follows:

$$\begin{cases} \sum_{k=1}^{\infty} A_k \cosh(\lambda_k H_r)\cos(\lambda_k z) = \sum_{l=1}^{\infty} D_l \sinh[\mu_l(L_a-H_r)]\cos(\mu_l z) + L(H_r) & (5.39a) \\ \sum_{k=1}^{\infty} A_k \lambda_k \sinh(\lambda_k H_r)\cos(\lambda_k z) = -\sum_{l=1}^{\infty} D_l \mu_l \cosh[\mu_l(L_a-H_r)]\cos(\mu_l z) + \beta^* & (5.39b) \end{cases}$$

for all $z \in [0, D_a-D_r]$

By multiplying (5.39a) and (5.39b) by $\cos\mu_i z$ and integrating over $z \in [0, D_a-D_r]$ we obtain:

$$\begin{cases} \sum_{k=1}^{\infty} A_k \cosh(\lambda_k H_r)\gamma_{ki} = \sum_{l=1}^{\infty} D_l \sinh[\mu_l(L_a-H_r)]\varphi_{li} + L(H_r)\cdot c_i & (5.40a) \\ \sum_{k=1}^{\infty} A_k \lambda_k \sinh(\lambda_k H_r)\gamma_{ki} = -\sum_{l=1}^{\infty} D_l \mu_l \cosh[\mu_l(L_a-H_r)]\varphi_{li} + \beta^* \cdot c_i & (5.40b) \end{cases}$$

for i = 1,2,...

where

$$c_i = \int_{0}^{D_a-D_r} \cos\mu_i z\,dz = \frac{1}{\mu_i}\sin[\mu_i(D_a-D_r)], \quad (i=1,2,\ldots). \tag{5.41}$$

Parameter β^* can be calculated from the total mass balance in the flow domain:

$$Q = Q_{bed} + Q_{bank} \tag{5.42}$$

$$Q = \int_o^{D_a} -k_a \frac{\partial\tilde{\phi}}{\partial x}|_{x=H_r} dz = -k_a \sum_{l=1}^{\infty} (-D_l\mu_l \cosh[\mu_l(L_a-H_r)] \int_o^{D_a} \cos\mu_l z\, dz) + \tag{5.43}$$

$$-k_a\beta^* D_a = -k_a\beta^* D_a$$

$$Q_{bed} \overset{(5.19)}{=} -k_a \sum_{k=1}^{\infty} A_k \sin[\lambda_k(D_a-D_r)] \cdot \sinh[\lambda_k H_r] \tag{5.44}$$

$$Q_{bank} = \int_{D_a-D_r}^{D_a} -k_a \frac{\partial\tilde{\phi}}{\partial x}|_{x=H_r} dx = -k_a \int_{D_a-D_r}^{D_a} \frac{\tilde{\phi}(H_r,z)}{\chi^2} dz =$$

$$= -\frac{k_a}{\chi^2} \int_{D_a-D_r}^{D_a} \left[\sum_{l=1}^{\infty} D_l \sinh[\mu_l(L_a-H_r)] \cos\mu_l z + \tilde{\phi}_o - \beta^*(L_a-H_r)\right] dz$$

$$= -\frac{k_a}{\chi^2} \left\{\sum_{l=1}^{\infty} D_l \sinh[\mu_l(L_a-H_r)] \int_{D_a-D_r}^{D_a} \cos\mu_l z\, dz + [\tilde{\phi}_o - \beta^*(L_a-H_r)] D_r\right\}$$

$$= -\frac{k_a}{\chi^2} \left\{\sum_{l=1}^{\infty} D_l \sinh[\mu_l(L_a-H_r)] \frac{1}{\mu_l} (\sin\mu_l D_d - \sin\mu_l(D_a-D_r)) + \right.$$

$$\left. + \left[\tilde{\phi}_o - \beta^*(L_a-H_r)\right] D_r \right\} =$$

$$= -\frac{k_a}{\chi^2}\left\{\sum_{l=1}^{\infty} D_l \sinh[\mu_l(L_a - H_r)]\left(-\frac{1}{\mu_l}\sin\mu_l(D_a - D_r)\right) + \right.$$

$$\left. + \left[\tilde{\phi}_o - \beta^*(L_a - H_r)\right]D_r\right\}.$$

Finally

$$Q_{bank} = -\frac{k_a}{\chi^2}\left\{-\sum_{l=1}^{\infty} D_l c_l \sinh[\mu_l(L_a - H_r)] + [\tilde{\phi}_o - \beta^*(L_a - H_r)]D_r\right\}. \quad (5.45)$$

After substituting all terms in the mass balance equation (5.42) we get:

$$-k_a\beta^* D_a = -k_a \sum_{k=1}^{\infty} A_k \sin[\lambda_k(D_a - D_r)]\sinh[\lambda_k H_r] +$$

$$+ \frac{k_a}{\chi^2}\sum_{l=1}^{\infty} D_l \sinh[\mu_l(L_a - H_r)]c_l - \frac{k_a}{\chi^2}[\tilde{\phi}_o - \beta^*(L_a - H_r)]D_r$$

From this

$$\begin{aligned}\beta^* &= \sum_{k=1}^{\infty} A_k \frac{\chi^2}{M}\sin[\lambda_k(D_a - D_r)]\sinh[\lambda_k H_r] + \\ &\quad - \sum_{l=1}^{\infty} D_l \frac{1}{M}\sinh[\mu_l(L_a - H_r)]c_l + \phi^*\end{aligned} \quad (5.46)$$

where

$$\phi^* = \frac{\tilde{\phi}_o D_r}{M} \quad (5.47)$$

$$M = D_a\chi^2 + (L_a - H_r)D_r. \quad (5.48)$$

After substituting (5.46) to (5.40b) we obtain the following set of algebraic equations:

$$\sum_{k=1}^{\infty} A_k \cosh(\lambda_k H_r)\gamma_{ki} - \sum_{l=1}^{\infty} D_l \sinh(\mu_l(L_a-H_r))\varphi_{li} = L(H_r)\cdot c_i$$

$$\sum_{k=1}^{\infty} A_k \sinh(\lambda_k H_r)\left\{\lambda_k\gamma_{ki} - \frac{\chi^2}{M}\sin(\lambda_k(D_a-D_r))c_i\right\} + \tag{5.49}$$

$$\sum_{l=1}^{\infty} D_l \left\{\mu_l \cosh[\mu_l(L_a-H_r)]\varphi_{li} + \frac{1}{M}\sinh[\mu_l(L_a-H_r)]c_l c_i\right\} = \phi^* c_i$$

$$; \ (i=1,2,...).$$

Knowing β^* we can calculate a term $L(H_r)$ in formula (5.49):

$$L(H_r) = \tilde{\phi}_o - \beta^*(L_a-H_r) =$$

$$= \tilde{\phi}_o - \sum_{k=1}^{\infty} A_k \frac{\chi^2(L_a-H_r)}{M}\sinh[\lambda_k H_r]\sin[\lambda_k(D_a-D_r)] + \tag{5.50}$$

$$+ \sum_{l=1}^{\infty} D_l \frac{(L_a-H_r)}{M} c_l \sinh[\mu_l(L_a-H_r)] - \phi^*(L_a-H_r)$$

Now, equations (5.48) can be rewritten in a matrix form:

$$\left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right] \left[\begin{array}{c} A_1 \\ A_2 \\ \vdots \\ \hline D_1 \\ D_2 \\ \vdots \end{array}\right] = \left[\begin{array}{c} \underline{b}_1 \\ \hline \underline{b}_2 \end{array}\right]$$

where

$$
\begin{cases}
A_{11}(i,k) = \cosh(\lambda_k H_r)\gamma_{ki} + \dfrac{\chi^2(L_a-H_r)}{M}\sinh[\lambda_k H_r]\sin[\lambda_k(D_a-D_r)]c_i \\
A_{12}(i,k) = -\sinh[\mu_k(L_a-H_r)]\left\{\varphi_{ki} + \dfrac{(L_a-H_r)}{M}c_k c_i\right\} \\
A_{21}(i,k) = \sinh(\lambda_k H_r)\left\{\lambda_k\gamma_{ki} - \dfrac{\chi^2}{M}\sin[\lambda_k(D_a-D_r)]c_i\right\} \\
A_{22}(i,k) = \mu_k\cosh[\mu_k(L_a-H_r)]\varphi_{ki} + \dfrac{1}{M}\sinh[\mu_k(L_a-H_r)]c_k c_i
\end{cases}
\quad , \quad (i,k = 1,2,...) \tag{5.52}
$$

and

$$
\begin{cases}
b_1(i) = [\tilde{\phi}_o - \phi^*(L_a-H_r)]\,c_i \\
b_2(i) = \phi^* c_i \quad , \qquad (i=1,2,...).
\end{cases} \tag{5.53}
$$

Remark 7: The hyperbolic sin and cosine in formulae (5.52) need some precalculations to avoid rising exponents. They can be represented as follows:

$$
\begin{cases}
\sinh\alpha = e^{\alpha}(1-e^{-2\alpha})/2 \quad \text{where } \alpha=\lambda_k H_r \\
\cosh\beta = e^{\beta}(1+e^{-2\beta})/2 \quad \text{where } \beta=\mu_k(L_a-H_r).
\end{cases}
$$

Factors e^{α} and e^{β} can be omitted when solving equations (5.51) but then the solutions $\tilde{A}_1, \tilde{A}_2, ...; \tilde{D}_1, \tilde{D}_2, ...$ must be interpreted as

$$
\begin{cases}
\tilde{A}_k = A_k e^{\alpha} \\
\tilde{D}_k = D_l e^{\beta}
\end{cases}
$$

Consequently, when calculating piezometric heads in regions I and II from formulae (5.18) and (5.31) one may write $\tilde{A}_k(1-\exp(-2\alpha))/2$ instead of

$A_k\sinh(\alpha)$ and $\tilde{D}_k(1-\exp(-2\beta))/2$ instead of $D_k\cosh(\beta)$ thus avoiding rising exponents once again.

An algorithm for calculating the solutions for the regions I and II can be summarized as follows:

i) Calculate A_k-s and D_l-s from (5.51)

ii) Calculate β^* from (5.46)

iii) Calculate ϕ^I from (5.18)

iv) Calculate ϕ^{II} from (5.32)

v) Check the mass balance for the flow region Ω using formulae (5.42)–(5.45).

5.2 Asymptotic behaviour of the solution

Case one: If the river is isolated from the aquifer, i.e. $c \to \infty$

$$c\to\infty \Rightarrow \chi^2\to\infty \overset{(5.50)}{\Rightarrow} \phi^*\to 0 \overset{(5.53)}{\Rightarrow} A_k\to 0 \Rightarrow \beta^*\to 0$$

$$A_k\to 0 \overset{(5.43)}{\Rightarrow} D_i\to 0 \Rightarrow \tilde{\phi}\to\tilde{\phi}_o \Rightarrow \phi\to\phi_o \text{ (in region II)}$$

Also in region I, $\phi \to \phi_o$. Indeed, $\chi^2\to\infty \Rightarrow \dfrac{1}{\chi^2}\to 0$

$$\overset{(5.15)}{\Rightarrow} \lambda\sin[\lambda(D_a-D_r)]\to 0 \Rightarrow \lambda\to 0 \overset{(5.11)}{\Rightarrow} \begin{Bmatrix} X(x)=\text{const} \\ Z(z)=\text{const} \end{Bmatrix} \overset{(5.14)}{\Rightarrow} \tilde{\phi}=\text{const.}$$

$$\overset{(5.38)}{\Rightarrow} \tilde{\phi}^I = \phi^{II} = L(H_r)\to\tilde{\phi}_o \Rightarrow (\tilde{\phi}\to\tilde{\phi}_o) \text{ everywhere, i.e. } \tilde{\phi} = \phi_o \text{ everywhere.}$$

Consequently, $Q = 0$

Case two If the river is in good contact with the aquifer, i.e. $c \to 0$

$$c\to 0 \Rightarrow \chi\to 0 \overset{(5.50)}{\Rightarrow} \phi^* \to \tilde{\phi}_o/(L_a-H_r)$$

$$\text{Also } \chi\to 0 \overset{(15)}{\Rightarrow} \tilde{\lambda}_k=\frac{2k-1}{2}\pi \Rightarrow \cos[\lambda_k(D_a-H_r)]\to 0$$

$\Rightarrow \phi(x, D_a - D_r) = \phi_r$ in region I.

In region II the solution has been chosen in such a way that it fulfils the third type b.c.) on Γ''_{12} i.e. $\chi^2 \frac{\partial \tilde{\phi}}{\partial x}|_{x=H_r} = \tilde{\phi}\,|_{x=H_r}$ for all $z \varepsilon [D_a - D_r\; D_a$.

When $\chi^2 \rightarrow 0 \;\Rightarrow\; \tilde{\phi}|_{x=H_r} \rightarrow 0 \;\Rightarrow\; \phi|_{x=H_r} = \phi_r$ for all $z \varepsilon [D_a - D_r D_a$

Consequently, when $c \rightarrow 0$, the solution obtained $\phi(x,z)$ is equal to ϕ_r along the bed and the bank of a river. Still $Q < Q_H$ for the reasons explained in Chapter 4.

<u>Case three</u> For the fully penetrating river, i.e. $D_r = D_a$ (or p=1)

$$D_r = D_a \overset{(5.41)}{\Rightarrow} c_l = 0 \text{ for } (l=1,2,...) \overset{(5.48)}{\Rightarrow} Q_{bank} = -\frac{k_a}{\chi^2}[\tilde{\phi}_o - \beta^*(L_a - H_r)]\,D_r .$$

Also $Q_{bed} = 0$ - see formula (5.47). Consequently, $Q = Q_{bank}$.

From (5.45) and (5.48) we have

$$-k_a \beta^* D_a = -\frac{k_a}{\chi^2}[\tilde{\phi}_o - \beta^*(L_a - H_r)]D_a \;\Rightarrow\; \beta^*[\chi^2 + (L_a - H_r)] = \tilde{\phi}_o$$

from which

$$\beta^* = \frac{\tilde{\phi}_o}{\chi^2 + (L_a - H_r)} \tag{5.54}$$

Formula (5.46) results in

$$Q = T_a \frac{\phi_r - \phi_o}{\chi^2 + (L_a - H_r)} \tag{5.55}$$

This is exactly the same result as that obtained for the 2D-horizontal flow model for the fully penetrating river - see formula (4.26).

Hence

$$Q = \tilde{Q}_H$$

It can also be easily shown that the piezometric head in region II changes linearly and is a function of x only. Indeed,

$$D_r = D_a \overset{(5.37)}{\Rightarrow} \gamma_{ki} = 0 \overset{(5.43)}{\Rightarrow} D_i = 0 \overset{(5.32)}{\Rightarrow} \tilde{\phi}(x,z) = L(x) = \tilde{\phi}_o - \beta^*(L_a - x)$$

with β^* given by (5.54).

6 Comparison of two-dimensional and three-dimensional models of the river-aquifer interaction - computer experiments

A number of computer experiments have been carried out to find out how the ratio between the exact total flow Q - formula (5.43) - and the two-dimensional total flow Q_h - formula (4.1) - depends on the hydraulic and geometric parameters of the river-aquifer system. The results of these experiments have been illustrated on the following Figures 6.1 - 6.9.

Figure 6.1 shows Q/Q_h as a function of the number of the infinite series terms (kmax) for number of the river penetration values (p) and for H_r = 5.0m, c = 1.0d

Figure 6.2 shows Q/Q_h as a function of penetration (p) of the river into aquifer for a number of hydraulic conductivity values (k_a) and for H_r = 5.0m, c = 1.0d

Figure 6.3 shows Q/Q_h as a function of hydraulic conductivity of the aquifer (k_a) for a number of river penetration values (p) and for H_r = 5.0m, c = 1.0d

Figure 6.4 shows Q/Q_h as a function of penetration (p) of the river into aquifer for a number of the river bed resistivity values (c) and for H_r = 5.0m, k_a = 10.0m/d

Figure 6.5 shows Q/Q_h as a function of the river bed resistivity (c) for a number of river penetration values (p) and for H_r = 5.0m, k_a = 10.0m/d

Figure 6.6 shows Q/Q_h as a function of penetration (p) of the river into aquifer for a number of river width values (H_r) and for k_a = 10.0m/d, c = 1.0d

Figure 6.7 shows Q/Q_h as a function of the river width (H_r) for a number of river penetration values (p) and for k_a = 10.0m/d, c = 1.0d

Figure 6.8 shows Q/Q_h as a function of the river bed resistivity (c) for a number of hydraulic conductivity values (k_a) and for H_r = 5.0m, p = 0.4

Figure 6.9 shows Q/Q_h as a function of hydraulic conductivity of the aquifer (k_a) for a number of the river bed resistivity values (c) and for H_r = 5.0m, p = 0.4

7 Conclusions

The results of the computer experiments show **in particular** that:

- with increasing penetration of the river into an aquifer the total 3D-flow becomes closer to the horizontal flow approximation (Q/Q_h becomes closer to 1.0) though for larger values of hydraulic conductivity k_a the 3D-flow exhibits values that are much smaller then 1.0

- with increasing values of the hydraulic conductivity ka the values of Q/Q_h generally decrease though when the river bed resistivity c is large (c of order of 50 days) an increase in k_a causes an increase in Q/Q_h

- for given penetration p and with increasing river bed resistivity c the ratio Q/Q_h increases

- for given penetration p and with increasing values of the river width H_r the ratio Q/Q_h decreases

- it was estimated that for most cases the number of terms that need to be summed up in order to obtain the convergence of the solution's infinite series is of order 30. There are however cases that causes problems in convergence of the series. In the case of p=0.5 all the orthogonal functions in coefficients c_i (see formula 5.41) are close to zero and thus the system of equations (5.40a, 5.40b) is close to being singular

The results also show several **common features** :

- the relationship between the Q/Q_h and the system parameters is nonlinear

- all the values of Q/Q_h are (as it was envisaged in paragraph 4.1.) less than 1.0

- all the hydraulic factors (like increasing k_a or decreasing c) that allow water from the river to penetrate deeper below the river bed into the aquifer cause an increasing difference between the two models (i.e. Q/Q_h decreasing)

- all the geometric factors (like increasing penetration p or decreasing width of the river H_r) cause water from the river to flow mostly through the river bank and then continuing almost horizontally into the aquifer. As the result the difference between the two models decreases (i.e. Q/Q_h increases)

Generally one may conclude that :

- 2D-horizontal model of groundwater flow always overestimates the total flow in the aquifer in the presence of the river-aquifer interaction

- since there is no clear (easy) relationship between the ratio Q/Q_h and parameters of the river-aquifer system it is more proper to use the full three-dimensional model for calculating total flow in the aquifer if the river-aquifer interaction needs to be taken into account

- at least two extensions of the 3D-flow model need to be developed for the river-aquifer interaction :

i) a model for the unconfined groundwater flow

ii) a model for more elaborate geometries of the river bad cross-section.

8 Acknowledgements

This report describes work carried out by Prof. Marek Nawalany from the Institute of Environmental Engineering - IEE (Warsaw University of Technology, Warsaw, Poland) during his two short visits at the HR Wallingford - in April 1992 and March 1993. The work was a part of research on the river-groundwater interaction carried out by the groundwater modelling group of Dr Charles Reeve within the Operations Department. The author would like to acknowledge the substantial help of his two young colleagues Mr Jakub Loch and Grzegorz Sinicyn (also from IEE) who contributed to the computational aspects of the project.

The author would like to express his thanks for both him and his colleagues for the financial support from HR Wallingford and the British Council.

Figures

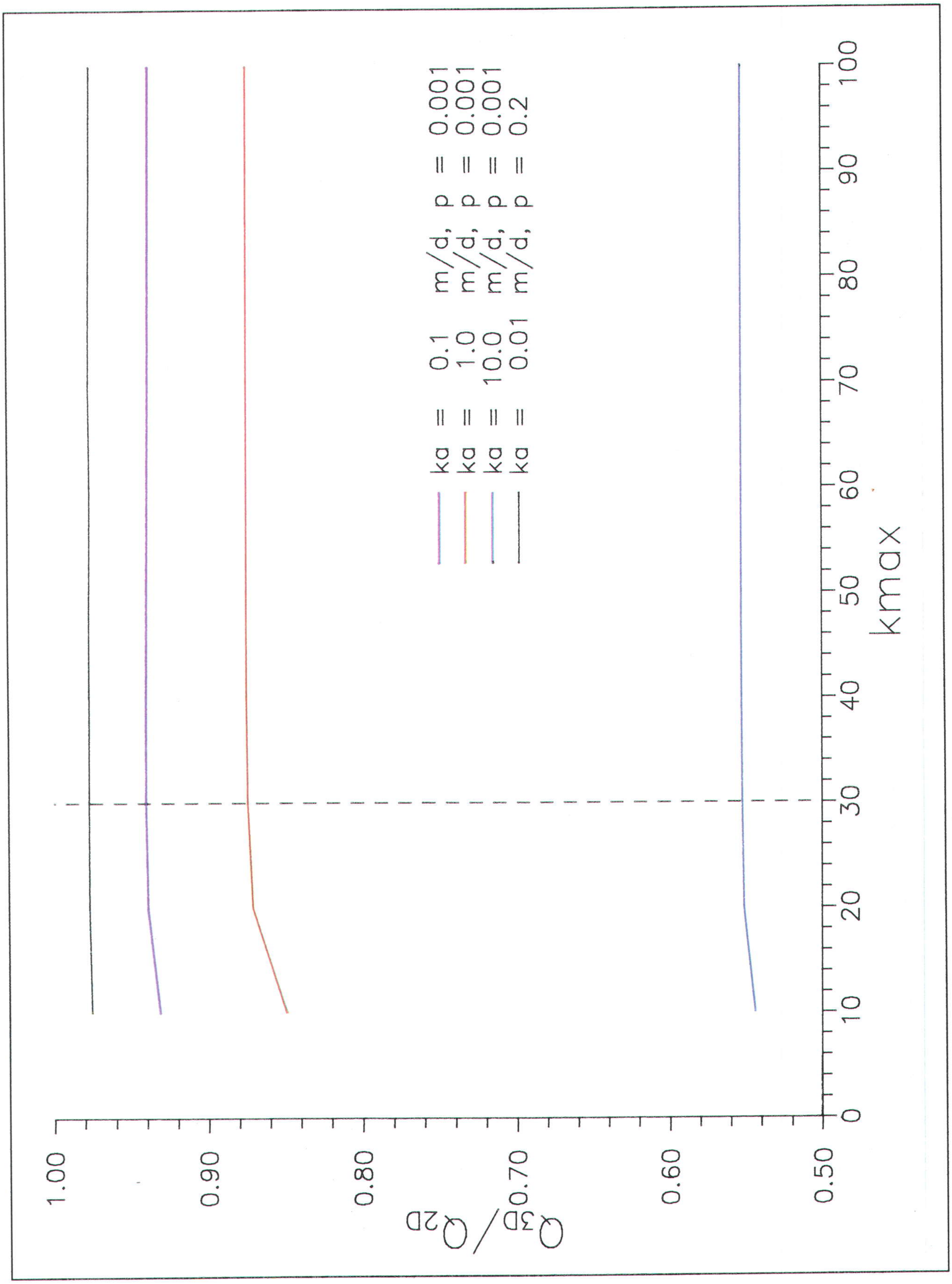

Figure 6.1 Convergence of the Q3D/Q2D as kmax increases

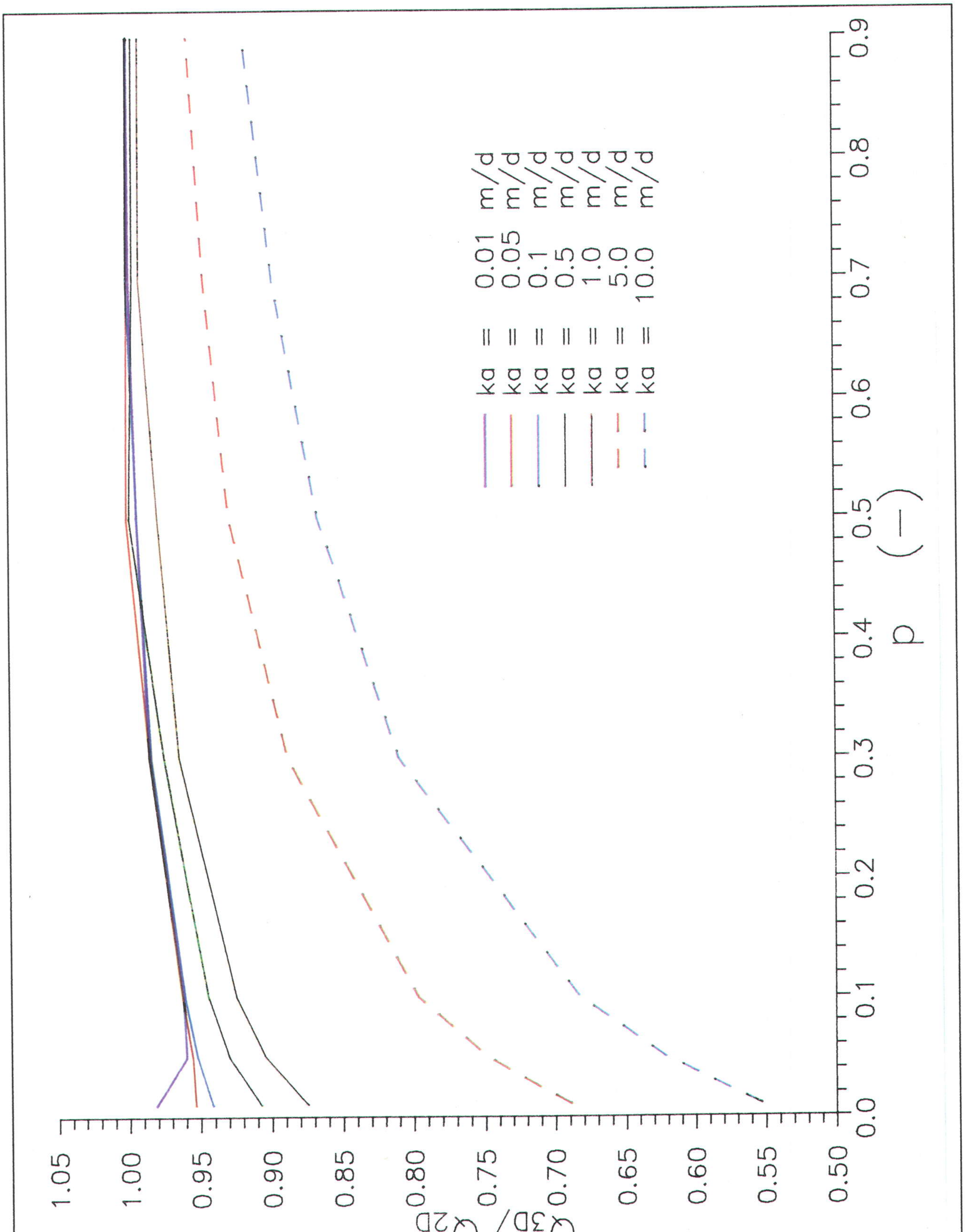

Figure 6.2 Dependence of Q3D/Q2D on penetration p (ka families)

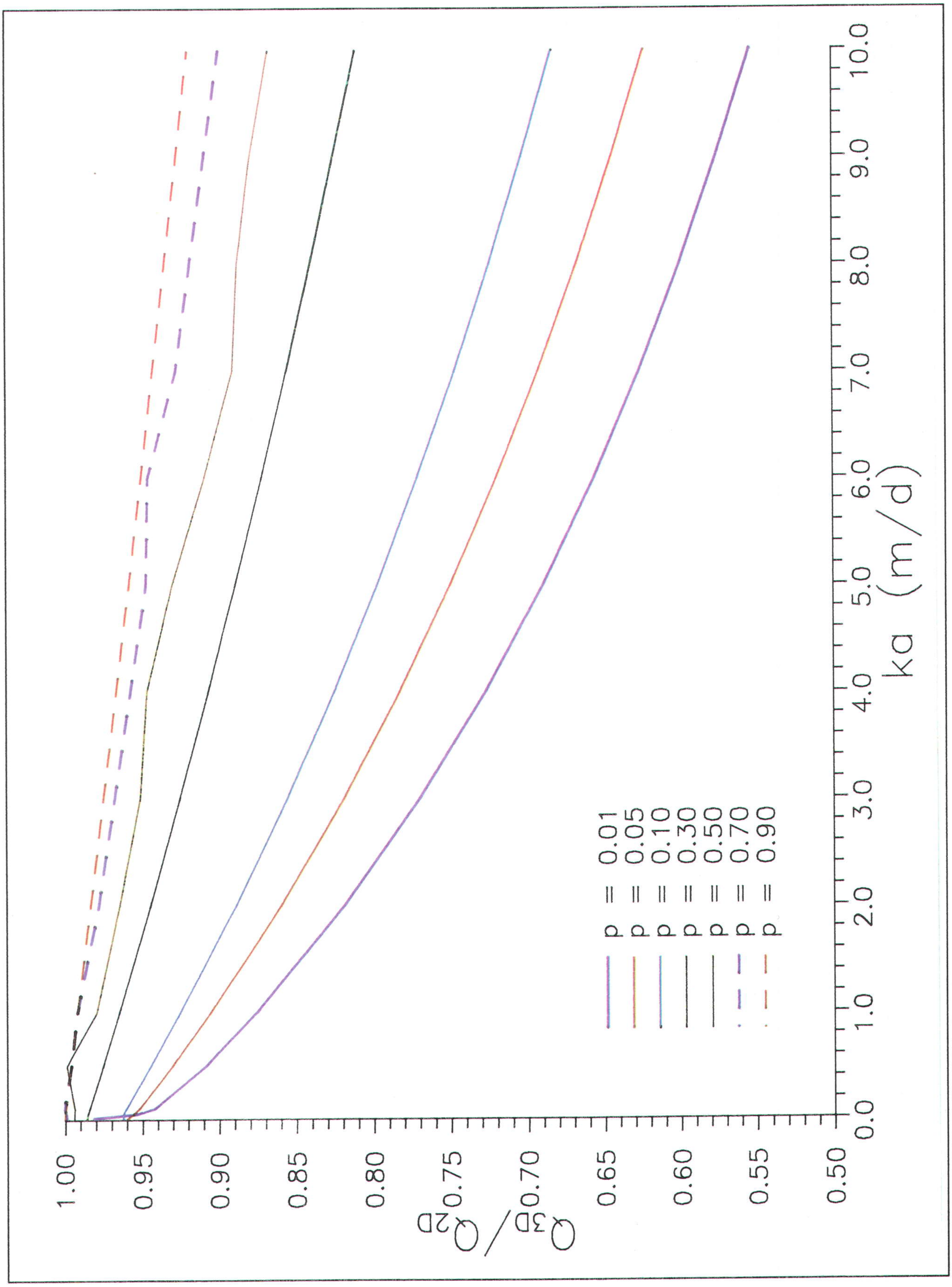

Figure 6.3 Dependence of Q3D/Q2D on hydraulic conductivity ka (p families)

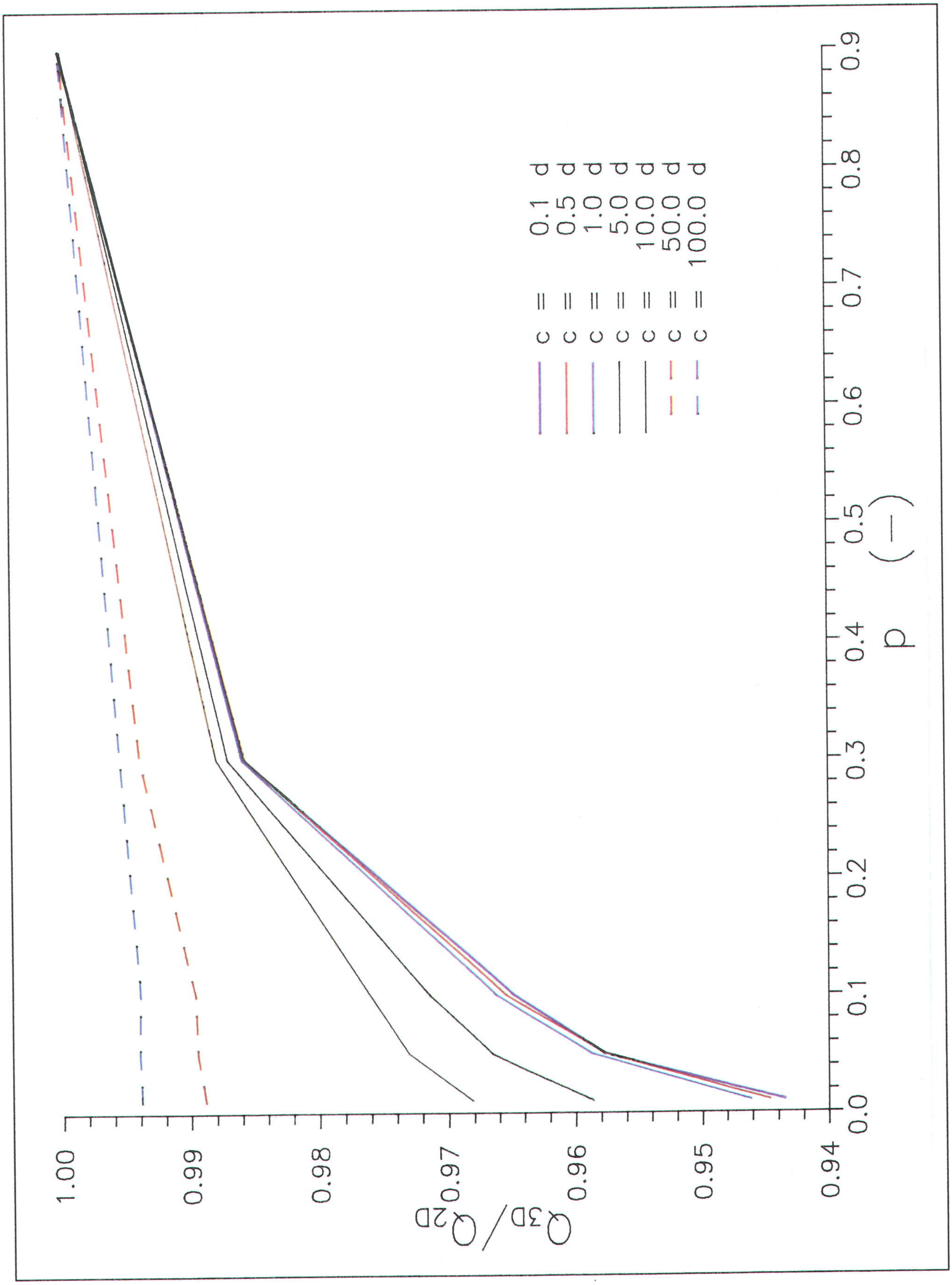

Figure 6.4 Dependence of Q3D/Q2D on penetration p (c families)

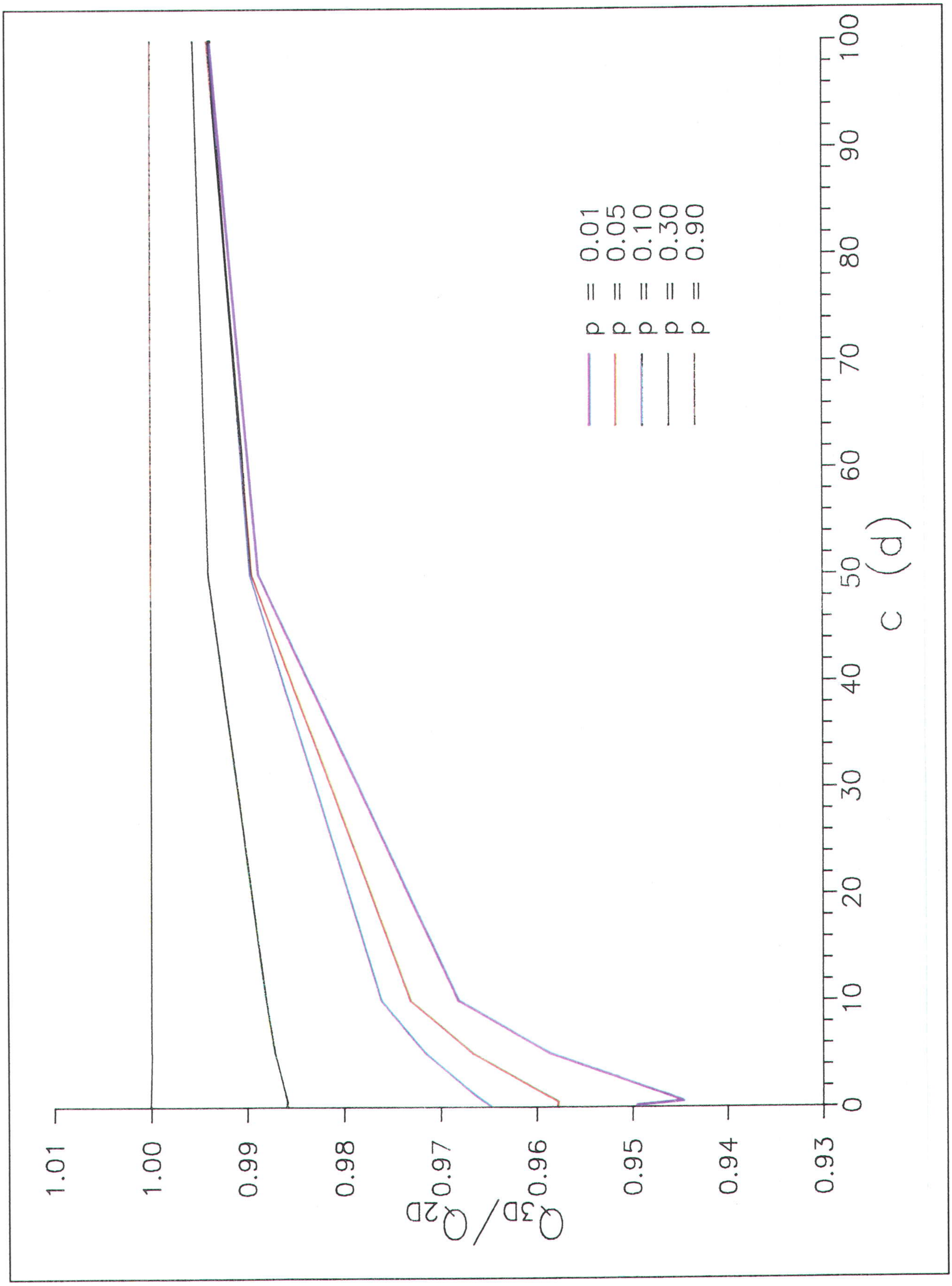

Figure 6.5 Dependence of Q3D/Q2D on the river bottom resistivity c (p families)

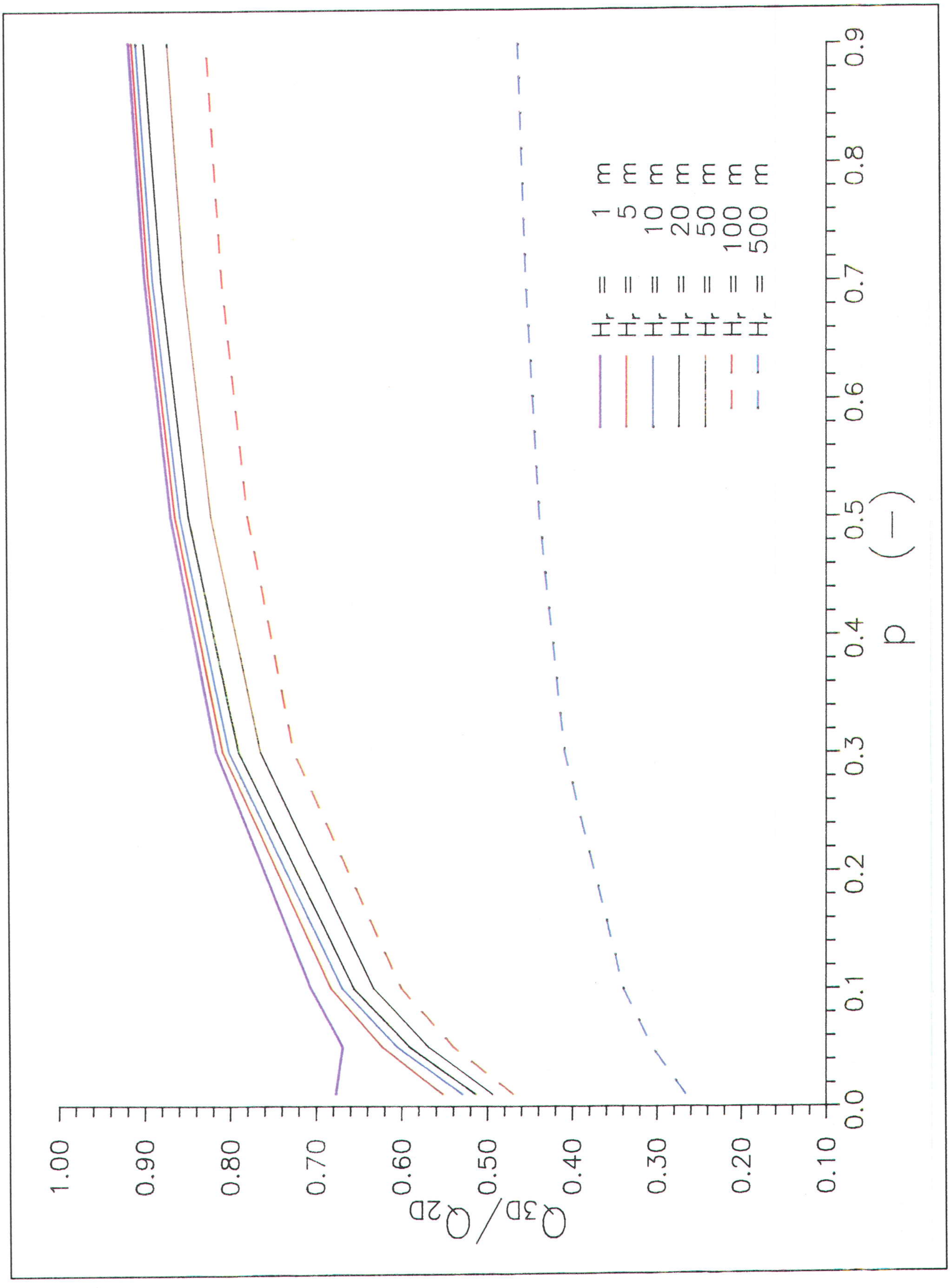

Figure 6.6 Dependence of Q3D/Q2D on penetration p (Hr families)

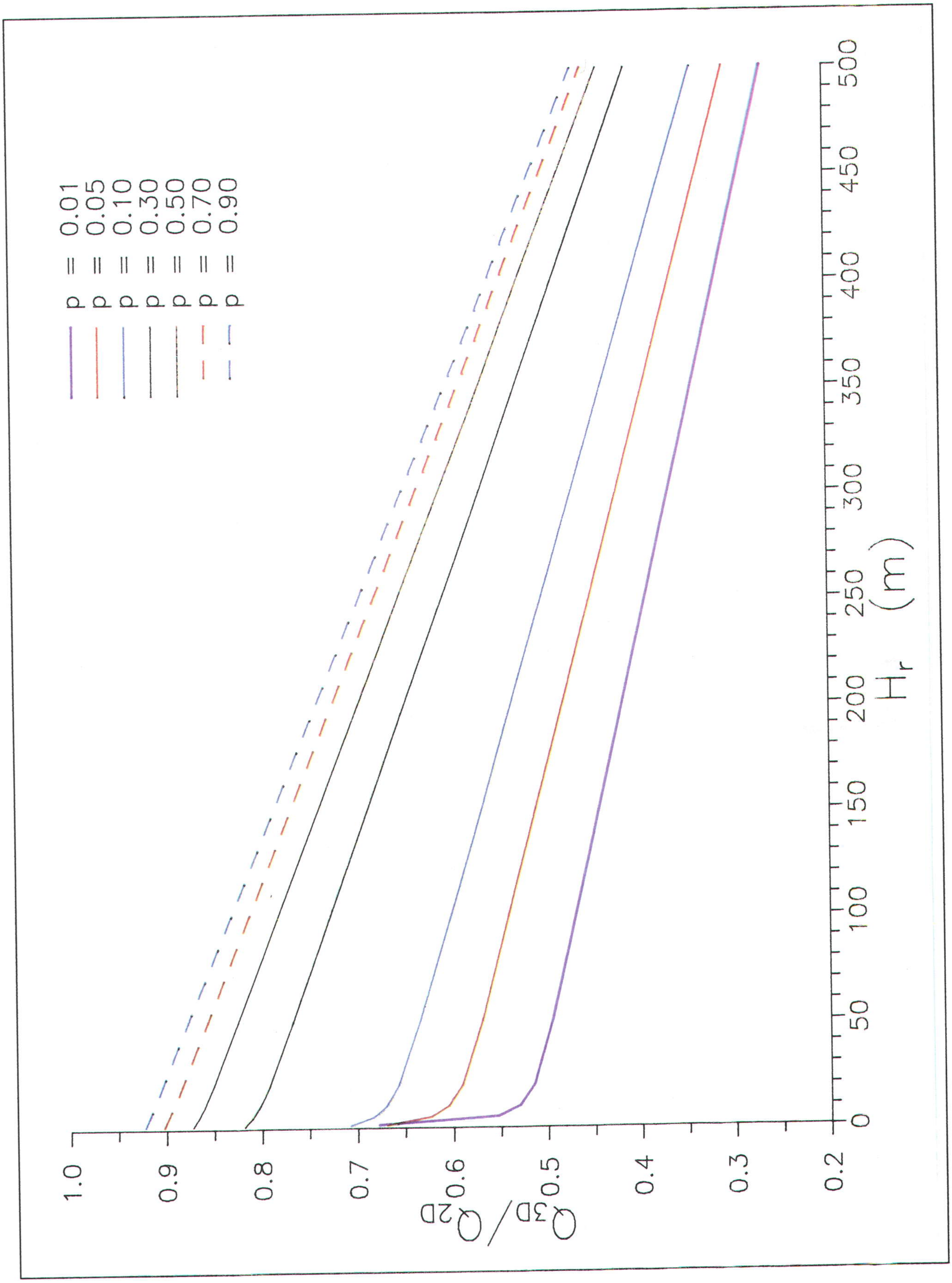

Figure 6.7 Dependence of Q3D/Q2D on the width aspect ratio Hr (p families)

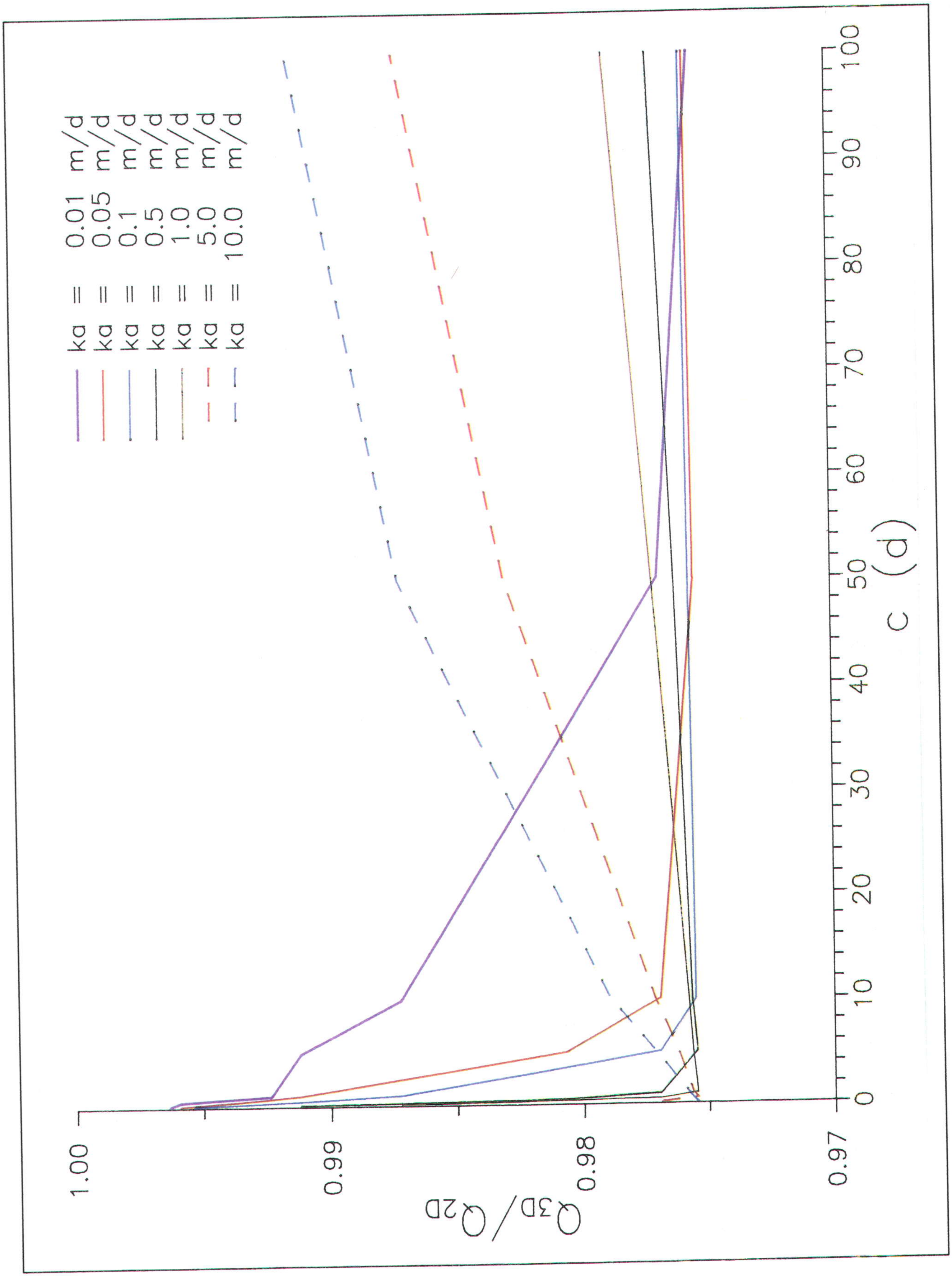

Figure 6.8 Dependence of Q3D/Q2D on the river bottom resistivity c (ka families)

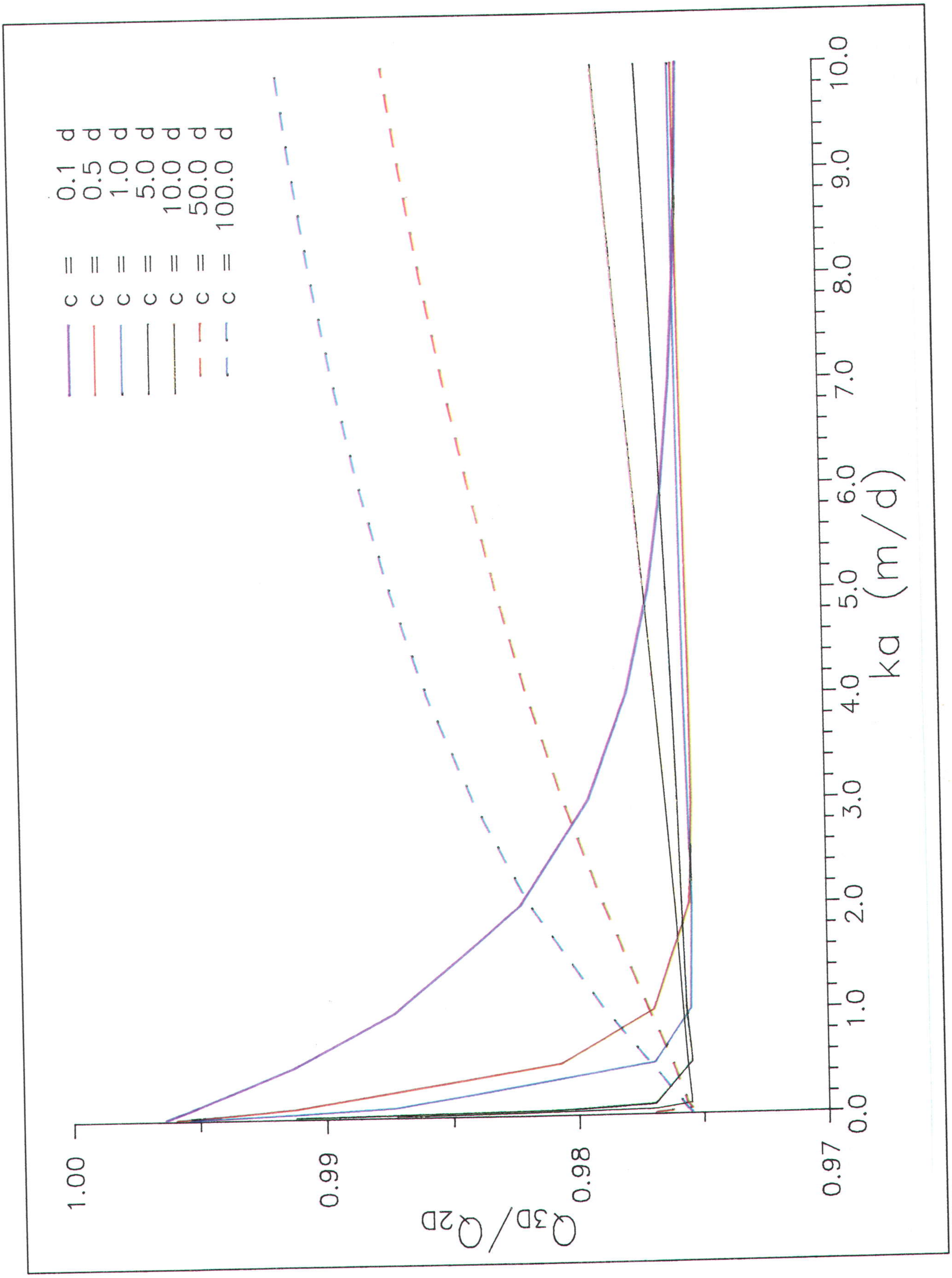

Figure 6.9 Dependence of Q3D/Q2D on the hydraulic conductivity ka (c families)