A One-Layer Model to Predict Development in Time of Static Armour

Anna Frangipane

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Registered Office: **HR Wallingford Ltd.** Howbery Park, Wallingford, Oxfordshire, OX10 8BA, UK
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Summary

The occurrence of static armour has been studied extensively from a number of different points of view.
Given the hydraulic conditions and the initial composition of the bed, many relations have been proposed, which predict the armoured distribution which develops.

There are fewer relations which predict the development of the armour layer in time and the changing in transport rate as the armour develops.
An attempt is made in this report to use a simple one layer model to predict the development of static armour and a comparison is presented of results obtained with the use of different transport formulae for non uniform sediments.

Notation

a	[L]	thickness of the mixing layer
a_0	[L]	initial thickness of the mixing layer
B	[L]	bed width
c	[-]	coefficient in eqs. (6) and (7)
c_i	[LT^{-1}]	celerity defined by eq. (41)
$(c_i)_{min}$	[LT^{-1}]	minimum value of celerity defined by eq. (41)
C	[-]	constant in eq. (8)
D	[L]	characteristic dimension of a uniform sediment
D_{avg}	[L]	average sediment size defined by Egiazaroff (1965)
D_i	[L]	average size of the i-th sediment class
D_g	[L]	geometric mean sediment dimension
D_{gr}	[-]	dimensionless parameter defined by eq. (B1)
D_m	[L]	mean sediment dimension
D_{max}	[L]	maximum size of the initial sediment mixture
D_{min}	[L]	minimum size of the initial sediment mixture
D_{nn}	[L]	sediment size such that nn% in weight of the original mixture is finer
D_u	[L]	scaling dimension for hiding functions
e_i	[-]	fraction of the i-th sediment class in the eroded material
F_{gr}	[-]	dimensionless parameter defined by eq. (B2)
f_i	[-]	fraction of the i-th sediment class in the armoured surface
G_{gr}	[-]	dimensionless parameter defined by eq. (B3)
h	[L]	flow depth
J	[-]	slope
LL	[L]	length of the flume
M	[-]	number of the movable sediment fractions in the mixture

Notation continued

m_i	[-]	fraction of the i-th sediment class in the exchange layer
N	[-]	number of sediment fractions in the mixture
p_i	[-]	fraction of the i-th sediment class in the parent bed material
p(A-V)	[-]	fraction obtained with an area by area procedure
p(V-V)	[-]	fraction obtained with a volume by volume procedure
q_s	$[MT^{-3}]$	total sediment transport rate per unit width by dry weight
q_{si}	$[MT^{-3}]$	sediment transport rate per unit width of the i-th sediment class, by dry weight
q'_s	$[MT^{-3}]$	total sediment transport rate per unit width, by immersed weight
$(q_s)_{vol}$	$[L^2T^{-1}]$	sediment transport volume rate per unit width
Re	[-]	grain Reynolds number; $Re=u_*D/v$
Rh	[L]	hydraulic radius
t	[T]	time
t_i	[-]	fraction of the i-th sediment class in the transported material
U	$[LT^{-1}]$	mean velocity of the flow
u_*	$[LT^{-1}]$	shear velocity; $u_*=(\tau/\varrho)^{0.5}$
x	[L]	distance along the flow
X	[-]	sediment concentration by dry weight $X=q_s/(\gamma q)$
X_E	[-]	non-dimensional function defined by Einstein (1950)
Y_E	[-]	non-dimensional function defined by Einstein (1950)
z	[L]	bed elevation
z_0	[L]	initial bed elevation
α	[-]	exponent in equation 24
γ	$[ML^{-2}T^{-2}]$	specific weight of water
γ_s	$[ML^{-2}T^{-2}]$	specific weight of sediment

Notation continued

Δt	[T]	time step
ε_{ci}	[-]	hiding function defined by (9)
ε_i	[-]	hiding function defined by (10)
Θ_c	[-]	entrainment parameter for a uniform material of characteristic size D; $\Theta_c=\tau_c/(\gamma_s-\gamma)D$
Θ_{ci}	[-]	entrainment parameter for a uniform material of characteristic size D_i; $\Theta_{ci}=\tau_{ci}/(\gamma_s-\gamma)D_i$
Θ_{cu}	[-]	entrainment parameter for a uniform material of characteristic size D_u; $\Theta_{cu}=\tau_{cu}/(\gamma_s-\gamma)D_u$
Θ_{cmi}	[-]	entrainment parameter in the sediment mixture for the i-th sediment class; $\Theta_{cmi}=\tau_{cmi}/(\gamma_s-\gamma)D_i$
Θ_{cmu}	[-]	entrainment parameter in the sediment mixture for the scaling size D_u; $\Theta_c=\tau_{cmu}/(\gamma_s-\gamma)D_u$
λ	[-]	bed porosity
λ_0	[-]	initial bed porosity
ν	$[L^2T]$	kinematic viscosity of water
ξ_i	[-]	hiding function defined by eq. (19)
ϱ	$[ML^3]$	density of the water
τ	$[ML^{-1}T^{-2}]$	mean applied shear stress
τ_i	$[ML^{-1}T^{-2}]$	corrected shear stress applied to the i-th sediment class of the mixture
τ_c	$[ML^{-1}T^{-2}]$	critical shear stress for a uniform material of characteristic size D
τ_{ci}	$[ML^{-1}T^{-2}]$	critical shear stress for a uniform material of characteristic size D_i
τ_{cu}	$[ML^{-1}T^{-2}]$	critical shear stress for a uniform material of characteristic size D_u
τ_{cmi}	$[ML^{-1}T^{-2}]$	critical shear stress in the mixture for the i-th sediment class
τ_{cmu}	$[ML^{-1}T^{-2}]$	critical shear stress in the mixture for the scaling size D_u

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1 Introduction

It has been widely reported that, when applying a flow to a non-uniform sediment, selective transport can lead to the formation of a coarser surface layer, through a process known as armouring.
This has been observed both experimentally and in the field. The process depends upon both the flow, as well as the composition of the sediment mixture and the transported material.
A state of equilibrium for the coarsened surface can result from the progressive decreasing of sediment transport (development of static armour), from the stability of the coarsest fraction and the mobility of the finest fractions (semi-static armour) or from the equalization in composition between the transported and the bed material (dynamic armour).
It has been observed that all size fractions of the original bed are present in an armoured surface.
Many attempts have been made to predict armour formation and to reproduce experimental and field data.
At different times a number of different aspects of the problem have been considered:

- sediment and hydraulic conditions under which armour develops;
- effects induced by the non uniformity of grains;
- the significance of the composition of the subsurface material;
- erosion associated with the development and temporal evolution of the surface
- and of the transport rate (Lamberti & Paris, 1992).

A number of procedures have been developed to predict the development of armour. A procedure is referred to as being a single-step method if it directly predicts the final equilibrium features from the initial conditions of the flow and sediment and is called a multi-step method if it follows the changes in the relevant variables, as the armoured surface develops in time (Sutherland, 1987).
Mathematical models, belonging to this second group, have become more and more complex in an attempt to improve the simulation of the observed exchanges of material between the flow and the bed and of the formation of the armoured surface.
While one-layer models confine the description of the phenomenon to the surface layer, analysing the changes in its composition due to the interaction with the flow, multi-layer models (Ribbernik, 1987; Di Silvio, 1991) introduce active sub-layers between the bottom surface and the lower undisturbed material and rigorously define the variations occurring between them.

1.1 Purpose and scope of the study

The present study was aimed at simulating the changes through time of the sediment transport rate and the final surface composition obtained experimentally for static armour conditions.
Similar works had been carried out before, i.e. by **Proffitt (1980)** and **Paris (1991)**, but the present study aimed to show that a simple one-layer model could reproduce the phenomenon observed in experimental results.
By comparing the results obtained from the adoption of different transport theories for non-uniform sediments it was hoped that conclusions could be drawn about the predictions of such transport theories and the use of hiding functions.

1.2 On the static armour formation

It has been reported that in a reach of non-uniform bed material exposed to a constant flow rate with no sediment input and such that all size of sediment are in motion, erosion causes an initial decrease in the slope of the bed (rotational degradation) followed, because of the changed value of the applied shear stresses, by a selective erosion (parallel degradation) leading to the stabilization of the armour surface.

In particular, downstream of the construction of a dam, the composition of the bed-surface gradually changes, as the finest fractions are eroded and not replaced by any upstream supply: the coarsened surface acts as a protective layer for the undisturbed underlying material till an increased flow disturbs the overlying armoured layer.

Experimentally static armour formation has been reproduced in flumes with a sediment-free input at constant discharge (Proffitt, 1980; Lamberti & Paris, 1992; Tait et al., 1992).

Analysing these results, it can be noticed that, for some runs, in a first short interval a great amount of material is removed at a nearly constant transport rate and that, successively, the transport rate gradually decreases.

Comparing the behaviour of the same mixture in a number of runs of differing flow rates, it can be noticed that this first phase is present only for high values of shear stress.

Low values of initial shear stress appear to be associated with an initially increasing sediment transport rate.

Proffitt (1980) observed that the initial phase is characterised by the formation of transverse coarse bars on the bed and that only afterwards armouring of the bed takes place with the tendency of the coarse particles to produce an imbricated bed.

He saw that the development of an armoured bed involves both the erosion of the finer particles and the re-arrangement of the coarser fractions.

He also remarked that, during the second phase a quasi-linear relation exists in a log-log plot between sediment transport rate and time. For all his runs the composition of the transported material remained nearly constant, agreeing with Gessler's (1970) theoretical statements, but in contrast with Tait et al.'s (1992) experimental results.

1.3 Brief review of existing one-layer models

An important aspect of the analysis of the coarsening of the bed surface is the definition of the thickness of the armoured layer.

At one time it was considered that changes in composition only occurred in the top most layer of sediment and did not affect underlying, undisturbed parent bed material.

Some researchers, however, have reported that changes in the bed composition are not only restricted to a layer that is only one grain thick, that is, to the material exposed to the flow, but that a finer sub-layer also forms under the coarser surface layer.

To reproduce these features theoretically, multi-layer model have been proposed, but in spite of their accuracy in armour description, they need the specification of a higher number of parameters than the existing one-layer models.

In this study the attention has been fixed on one-layer models.

Hirano (1971) seems to have presented the first one-layer model. He examined both degradation and deposition in a reach and, assuming that the changes of the characters of the material is restricted within a layer of variable thickness on the bed surface (exchange layer), he imposed, for the erosion condition, material coming from the undisturbed underlying mixture to feed it.

For the development of armouring, if the porosity of the bed is treated as a constant during degradation, that is, $\lambda(x,t) = \lambda_0$, the general formulation of the continuity equation, referred to volumes and applied to a constant width reach, becomes:

$$\frac{\partial z}{\partial t} = - \frac{1}{1-\lambda_0} \frac{\partial (q_s)_{vol}}{\partial x} , \tag{1}$$

where z is the bed elevation, t the time, B the bed width, $(q_s)_{vol}$ the volume rate of sediment transport per unit width and x the distance along the flow. This is the sum of the continuity equations derived for each grain size class:

$$\frac{\partial m_i}{\partial t} = \frac{1}{a} (t_i - p_i) \frac{\partial z}{\partial t} - \frac{(q_s)_{vol}}{a(1-\lambda_0)} \frac{\partial t_i}{\partial x} , \tag{2}$$

where p_i, m_i and t_i are the fraction of the i-th sediment class, with average representative dimension D_i, in the original material, in the exchange layer of thickness a and in movement, respectively.

If the porosity of the sediment is not considered, the continuity equation becomes:

$$\frac{\partial z}{\partial t} = - \frac{\partial (q_s)_{vol}}{\partial x} , \tag{3}$$

which can be written, for each grain size class, as:

$$\frac{\partial m_i}{\partial t} = \frac{1}{a} (t_i - p_i) \frac{\partial z}{\partial t} - \frac{(q_s)_{vol}}{a} \frac{\partial t_i}{\partial x} . \tag{4}$$

These equations correspond to the equations:

$$\frac{\partial a m_i}{\partial t} + p_i \frac{\partial (z-a)}{\partial t} + \frac{\partial (q_s)_{vol} t_i}{\partial x} = 0 , \tag{5}$$

in **Lamberti & Paris (1992)**, with the exception that a is treated as a variable.

To solve the problem, if $(q_s)_{vol} t_i$ is expressed as a function of m_i, a fixed thickness for the active layer must be introduced. Hirano used a value close to the maximum grain size (D_{max}) of the original mixture. Other authors suggested values ranging from D_{50} to D_{max} (D_{nn} represents the dimension such that nn% in weight of the mixture is finer).
To fit experimental results, a particular solution of the general problem expressed by the above system of equations must be found through the introduction of appropriate initial and boundary conditions.
Unfortunately, the method used by Hirano to express them is not clear.

Ashida & Michiue (1971), on a control volume fitting the entire flume, imposed, for an active layer of fixed thickness, the replacement of the eroded sediment by a same weight of material having the composition as the active layer.
This approach led to the equation:

$$m_i(t+\Delta t) = (1+c)\cdot m_i(t) - c\cdot e_i(t) \ , \qquad (6)$$

where Δt is the time step, c is the fraction of the active layer eroded at each time step and e_i the fraction of the i-th class in the eroded material.
The size distribution of the eroded sediment is found by applying a transport rate equation to the surface distribution.
The solution of the model is obtained treating c as a constant.
This implies that, as it is proportional to the product $(q_s)_{vol}\cdot\Delta t$, during the development of the armour layer longer time steps were adopted, as the value of the transport rate decreased with time.
The stability of the solution depends on the value of the constant c.

An approach similar to that of Ashida & Michiue was presented by **Bayazit (1975).**

Proffitt (1980), using the same control volume as that used by Ashida & Michiue, imposed a balance in weight between the eroded material and the sediment supplied from the parent bed, for an active layer of fixed thickness, and obtained the equation:

$$m_i(t+\Delta t) = m_i(t) - c\cdot e_i(t) + c\cdot p_i \ . \qquad (7)$$

The main difference between the two formulae is that in this case the eroded material is replaced by the same weight of material coming from the undisturbed underlying bed, while Ashida & Michiue assumed that the replacement material has the same composition as that of the active layer.

2 *Hiding functions*

The calculation of the sediment transport of a graded sediment can be carried out utilising a formula conceived for uniform sediments.
In such an approach, the total transported material (q_s) is related to the sediment transport of each grain size class of the mixture, thought of as a uniform material, (q_{si}), through the relation:

$$q_s = \Sigma_i \, q_{si}\cdot p_i \ , \qquad (8)$$

where p_i is the fraction of the i-th sediment class, with average representative dimension D_i, in the original material.

Comparison with experimental results shows that the behaviour of the coarsest and finest fractions is not exactly reproduced by such an approach.
Results show that they have, respectively, greater and lower mobilities than predicted, both in terms of the threshold of motion and for transporting conditions.
The complex interactions between the flow and the mixture, which result in this response, are still to be completely understood, but must include both static

(e.g.: sheltering of the finest fractions and protrusion of the coarsest ones) and dynamic (e.g.: turbulences and collisions) aspects.
To overcome these difficulties, analytical and empirical functions have been introduced. These are normally referred to as **hiding functions** and are used to modify the threshold conditions of motion and sediment transport formulae for a uniform bed material, adapting them to the case of graded sediments.
A complete outline of the problem, paying attention to recent developments, can be found in **Sutherland (1991)**.

2.1 General description

Threshold of motion
Indicating with τ_c the mean applied shear stress at the threshold of motion, or critical shear stress, for a uniform material with grain size D and specific weight γ_s, it is well known that the corresponding entrainment parameter Θ_c ($\Theta_c = \tau_c/(\gamma_s-\gamma)D$), where γ is the specific weight of the water) is a function of the Reynolds number of the grain ($Re = u_*D/\nu$, where u_* is the shear velocity of the flow and ν is the kinematic viscosity of the water).
Given a uniform material and flow parameters, the value of Θ_c can be obtained using the Shields' diagram, which relates Θ_c to Re, or from one of the existing sediment transport formulae for the condition of initiation of motion.

In a mixture, the value of the entrainment parameter at threshold conditions for each grain size fraction, $\Theta_{cmi} = \tau_{cmi}/(\gamma_s-\gamma)D_i$ -where τ_{cmi} is the critical shear stress of that fraction in the mixture -can be related to that of a uniform material of the same size, $\Theta_i = \tau_{ci}/(\gamma_s-\gamma)D_i$, using a hiding function, ε_{ci}, defined as:

$$\Theta_{cmi} = \varepsilon_{ci} \cdot \Theta_{ci} \ . \tag{9}$$

Since the coarsest fractions in a mixture are less difficult to move than the same size in a uniform sized sediment, it follows that for the coarsest fractions $\varepsilon_{ci}<1$.
Correspondingly the finest fractions are less easily to move and so for these sizes the value of $\varepsilon_{ci}>1$.
Between these two extremes there is a sediment size D_u, for which no correction is required, that is $\Theta_{cmu}=\Theta_{ci}$, $\varepsilon_{ci}=1$.
The subscript "u" is used to denote uniform behaviour.
So, for $D_i>D_u$, $\Theta_{cmi}<\Theta_{ci}$, while for $D_i<D_u$ we have $\Theta_{cmi}>\Theta_{ci}$: that is the coarser fractions move for lower values of the applied shear stress than in a uniform bed, while finer fractions move at a higher applied shear stress.
In the most general way, the threshold conditions in a given mixture can be represented, in the Θ_c-Re plane, by a deformed Shields' curve, which intersects the Shields' curve for uniform sediment at the point $P \equiv (u_*D_u/\nu, \Theta_{cu}) \equiv (u_*D_u/\nu, \Theta_{cmu})$ as in figure 1.
The degree of deformation away from the Shields' curve depends upon the degree of non-uniformity of the sediment mixture. As the mixture becomes more uniform, the curve approaches more closely that for uniform sediment and also their intersection point moves to the one representing the conditions of uniform sediment.

Referring to D_u, a second hiding function, ε_i, can be defined:

$$\Theta_{cmi} = \varepsilon_i \cdot \Theta_{cu} \ . \tag{10}$$

The two functions are linked by the relation:

$$\varepsilon_i = \varepsilon_{ci} \cdot \frac{\Theta_{ci}}{\Theta_{cu}} . \tag{11}$$

If we only consider sediments that are sufficiently large so that the critical shear stress is constant with grain size, then $\Theta_{ci}=\Theta_{cu}=\Theta_{cmu}$ for every size and equation (11) becomes:

$$\varepsilon_i = \varepsilon_{ci} ; \tag{12}$$

in this case it can be stated that for $D_i>D_u$ is $\Theta_{cmi}<\Theta_{cu}$ and for $D_i<D_u$ is $\Theta_{cmi}>\Theta_{cu}$.
This occurrence can be expressed through the relation:

$$\varepsilon_i = (\frac{D_i}{D_u})^{-b} , \tag{13}$$

(see Figure 2), with b=0 when no hiding effects are present.

In this circumstance:

$$\varepsilon_i = 1, \text{ as } \Theta_{cmi}=\Theta_{cu}=\Theta_{ci} . \tag{14}$$

Equations (12) to (14) differ to those presented by **Sutherland (1991)** for the assumption that the value of Θ_{ci} is the same for all the fraction in the mixture. It can be verified that, when dealing with sediment mixtures with a variable uniform sediment threshold parameter, they are not exact any more.

The concept of **equal mobility** of the grains is often mentioned and in the context of threshold of motion this implies a situation in which all the fractions in a mixture possess the same critical shear stress:

$$\tau_{cmi} = \tau_{cu} = \text{CONSTANT} \tag{15}$$

This means that :

$$\Theta_{ci} = \frac{\tau_{cmi}}{(\gamma_s-\gamma)D_i} = \frac{D_u}{D_i} \cdot \frac{\tau_{cu}}{(\gamma_s-\gamma)D_u} = \frac{D_u}{D_i} \Theta_{cu} , \tag{16}$$

that is:

$$\varepsilon_i = \frac{D_u}{D_i} \tag{17}$$

and, from equation (11):

$$\varepsilon_{ci} = \frac{\Theta_{cu}}{\Theta_{ci}} = \frac{D_u}{D_i} \cdot \frac{\Theta_{cu}}{\Theta_{ci}} = \frac{\tau_{cu}}{\tau_{ci}} \ . \tag{18}$$

Sediment transporting conditions

Until now just the threshold of motion has been considered, but the interactions between the grains of the mixture are also important when sediment transport is taking place.

Many relations have been proposed to evaluate the sediment transport rate for a flow transporting both bed load and suspended material.

In the range of interest of the present studies on static armour formation, the total sediment transport can be identified with the former, whose features, such as saltation and rolling, are strictly linked with the composition of the bed surface.

A way to describe such interactions is to use a hiding function (ξ_i) that introduces a modification in the applied shear stress for each grain size class (τ_i), compared to the mean shear stress on the bed (τ):

$$\tau_i = \xi_i \cdot \tau \ . \tag{19}$$

Depending upon the formula adopted, the effects of hiding for each grain size class can be represented by a modification to the applied shear stress, through ξ_i, or to the threshold value, through ε_i or ε_{ci}, in comparison with the uniform bed conditions.

As they have been conceived in different ways, generally speaking, there is no link between the three types of hiding function, except the relationship, expressed by equation (11), given above.

As a result, comparisons are very difficult to make.

For sediment transport formulae in which, for each grain size, the effective applied shear stress and the critical shear stress only appear in the form τ_i/τ_{cmi}, the correction that is introduced can be regarded either as a ξ_i or a ε_{ci} type hiding function (Ribbernik, 1987).

In the past authors have proposed different forms of expressions for hiding functions, see **Sutherland (1991)** and **Ribbernik (1989)**, in an attempt to correct the results given by a specific sediment transport formula.

It is important to notice that, when applying hiding functions derived for sediment transport equation one formula to a different one, for example, because of the lack of a specific one or because of the complexity of the appropriate formula, it should be remembered that some of them have been conceived in the context of a specific formula, for example Einstein's, and they are not adaptable to different sediment transport equations.

2.2 Brief review

A very basic review of some sediment transport formula and of their hiding functions is now presented.

Egiazaroff (1957) presented a sediment transport formula for uniform material in a polynomial form:

$$\frac{q'_s}{\gamma . q} = K \left(\frac{\tau/(\gamma_s-\gamma)D}{\Theta_c} - 1 \right) J^{0.5} , \tag{20}$$

where q'_s is the sediment transport rate by immersed weight, q is the flow rate and K is a coefficient, depending on the shape of the grains, whose value is 0.015 for the sphericity characteristic of sand.
For complete turbulence a value of $\Theta_c = 0.06$ is assumed.
In 1965 he introduced a correction to this function, based on experimental data, to allow its use for a widely graded sediment.
Basically, for the case of a complete turbulence, the entrainment parameter at threshold, equal for all the sizes - $\Theta_{ci} = 0.06$ - is substituted by the expression:

$$\Theta_{cmi} = \Theta_{ci} \cdot \left(\frac{Log_{10}19}{Log_{10}(19D_i/D_{avg})} \right)^2 , \tag{21}$$

where the D_{avg} value is obtained by the combination of the average dimensions of the grain in movement and in the bed and is the size for which no correction is needed.
The expression was proposed for D_i/D_{avg} ranging between 0.4 and 8.0.
The ratio of the logarithmic terms can be seen as a hiding function $\varepsilon_i = \varepsilon_{is}$, depending on the sediment distribution through D_{avg}, and on the size being considered.
This hiding function has been widely used, overcoming the problem in evaluating D_{avg} by using instead the values of D_g, the geometric mean diameter, or D_m, the mean diameter, or D_{50} of the bed distribution.

Ashida & Michiue (1971) applied this function to a formula of the Meyer-Peter Müller type (see Appendix A), using a wider range of variation for D_i/D_{avg}.
Calibration with experimental data suggested the correction:

$$\varepsilon_i = 0.85 \cdot \frac{D_{avg}}{D_i} \quad \text{if } D_i/D_{avg} < 0.4 . \tag{22}$$

Day (1980), fitting experimental results, proposed a hiding function for the Ackers & White (1973) sediment transport formula (see Appendix B).
This formula expresses the sediment load as a function of the sediment mobility. Day modified the equation to account for non-uniform sediments by changing the threshold value of the sediment mobility.
The hiding function used is of the ε_{ci} type and depends on the distribution of the mixture through a scaling size D_u, for which no correction is needed. The hiding function depends on the characteristic diameters D_{16}, D_{50}, D_{84} and of the considered size.

On the basis of Proffitt (1980) experimental data, a hiding function was proposed by **Proffitt & Sutherland (1983)** for the Ackers & White formula in the ξ_i form, defined by equation (19). This formula allowed a correction to the mean applied shear stress for each grain size class; it depends on the size analysed and on a scaling size, which are functions both of the mixture and of the flow via the entrainment parameter $\Theta_{50} = \tau/(\gamma_s-\gamma)D_{50}$.

Even if beyond the purposes of this work a brief mention of the work by **Einstein (1950)** is necessary.
He first introduced the concept of hiding and translated it in his stochastic theory for uniform sediment into a ξ_i function correcting the shear stress applied to the finer sizes of the mixture. No consideration was given to the impact on the coarser fractions.
To convert his transport formula to the graded case, he introduced also two more non-dimensional functions, X_E and Y_E. Unfortunately it is very difficult to compare his hiding function with other ones of the same type.
In time, the fitting of new experimental data to the extended Einstein's formula has shown that the ξ_i functions give an excessive correction to the finest fractions. This lead **Einstein & Chien (1953)** and, later, **Pemberton (1972)** to propose a smoothed function ξ_i.

2.3 Adopted functions

In the present work both **Meyer-Peter & Müller (1948)** and **Ackers & White (1973)** sediment transport formulae are introduced in the model.
First they are used in their original formulation, just correcting the results through the availability in the bed, but later hiding function are adopted.
The **Egiazaroff (1965)** function, as corrected by **Ashida & Michiue (1971)**, is associated to Meyer-Peter & Müller formula, with the reference size D_{avg} calculated as D_g, the geometric mean dimension of the active layer.
The Ackers & White formula has been modified by using **Day (1980)** hiding function.

3 Experimental data

Many reports of experiments on static armour formation can be found in the literature.
Unfortunately few of these present detailed data on the time dependence of the main variables in the development of the armour layer and, in particular, the values of total transported material and its distribution through time.

3.1 Proffitt (1980)

A tilting flume, with painted walls, 22 m long and 0.605 m wide, was used. Recirculating clear water was supplied at a controlled temperature.
The slopes used ranged between 0.0029 and 0.0043.
Under nearly constant flow conditions, an armour coat was produced by parallel degradation in 18 runs for 4 different sediment mixtures which were very close to a log-normal distribution. The sediment sizes ranged from a D_{min} = 0.075 mm to a D_{max} = 38.10 mm.
The duration of the runs was between 24 and 98 hours.
One of the runs was stopped at intervals to take photographs of the different phases of the armour formation.
The author presented the plots reproducing the time dependence of total transport rate and of the eroded material.
Values of the initial and the armoured distributions are also available, as well as data on the hydraulic conditions.

3.2 Tait et al. (1992)

Data were obtained from 4 laboratory experiments on static armour formation carried out in a glass-sided recirculating tilting flume which was 12.5 m long and 0.3 wide.
The slopes used ranged between 0.001 and 0.004.
The hydraulic conditions were kept nearly constant and the armour layers were produced by parallel degradation using 3 different mixtures with size in the range of 0.05 and 7.0 mm.
The duration of the runs was between 50 and 100 hours.
The evolution of the armoured surface was followed by analysing photographs, taken every 6 hours, along the flume.
Diagrams of the initial mixtures, the bed load and the armoured surface variations for some cases and evolution in time of sediment transport rate are presented.
In Tait (1992) the changes in time of the bedload composition are reported for all the runs.

3.3 Notes on the sediment sampling and analysis

Attention must be paid in reading the experimental armour layer distribution curve, since it has been noticed that mistakes can be caused by confusing different procedures in sampling and analysis, see Proffitt (1980).
Areal and volume sampling can be chosen to obtain a sample which is representative of the mixture and the obtained sample can be analysed both by area or by volume.
The initial sediment composing the bed is generally sampled by bulk volume and analysed by weight.
With the assumption of constant specific weight and porosity, this procedure is a classic one of volume by volume.
While area by area and volume by volume analysis should give, for a homogeneously distributed mixture, the same results, a relation is needed to pass from an area by volume, that is the case of an armour layer sampled areally and analyzed by weighting, to a volume by volume analysis.
From the relation proposed by Kellerhals & Bray (1971) for an ideal mixture composed of cubes of 3 different sizes.

$$p(V\text{-}V)_i = C \cdot p(A\text{-}V)_i \, D_i^{-1} \,, \tag{23}$$

where $p(V\text{-}V)_i$ and $p(A\text{-}V)_i$ are the fractions obtained volume by volume and area by area for the i-th sieve size, C is a constant equal to the total sample weight divided by the total adjusted weight and D_i is the i-th size, Proffitt (1980) derived the following equation for the natural mixtures that he used:

$$p(V\text{-}V)_i = C \cdot p(A\text{-}V)_i \, D_i^{-(\alpha)},$$

where α is in the range 0.4 to 0.54. (24)

Such a conversion should be adopted to compare the results that a model gives for the armour layer distribution, generally presented as volume by volume, with the experimental data available, about always, as area by volume.

4 Description of the model

The proposed model was developed from the scheme introduced by Hirano (1971), adapting it to match experimental conditions, that is an initial bed consisting of a known non-uniform sediment and no sediment inflow at the upstream end of the flume.
In the model different assumptions were tested to check the influence on the results of the characterization of the exchange - or mixing - layer.
The solution to the differential equations describing the model depends both on these assumptions and on the boundary and initial conditions chosen.

4.1 General formulation

As previously reported, Hirano (1971) first proposed a formulation of a one-layer model.

For steady hydraulic conditions, the principle of continuity, applied to a control volume of width B and length dx, during an interval dt, provides to the most general equation:

$$\frac{\partial z}{\partial t} = -\left(\frac{1}{(1-\lambda_0)B}\,\frac{\partial(B(q_s)_{vol})}{\partial x} + \frac{a}{1-\lambda_0}\,\frac{\partial \lambda}{\partial t} + \frac{\lambda-\lambda_0}{1-\lambda_0}\,\frac{\partial a}{\partial t}\right), \tag{25}$$

where z is the bed elevation, $(q_s)_{vol}$ the volume rate of sediment transport for unit width, λ the porosity of the bed, with initial value λ_0, a the thickness of the exchange layer, t the time and x the distance along the flow.

For each grain size class of the mixture, the continuity equation takes the form:

$$\frac{\partial m_i}{\partial t} = \frac{(t_i - p_i)}{a}\,\frac{\partial z}{\partial t} - \frac{(q_s)_{vol}}{a(1-\lambda)}\,\frac{\partial t_i}{\partial x} + (m_i - p_i)\left(\frac{1}{(1-\lambda)}\,\frac{\partial \lambda}{\partial t} - \frac{1}{a}\,\frac{\partial a}{\partial t}\right), \tag{26}$$

for i = 1 to N, where N is the number of the sediment classes.
In equations (26) p_i, $m_i(x,t)$ and $t_i(x,t)$ are the fractions of the i-th sediment class, with average representative dimension D_i, in the original material, in the exchange layer and in movement, respectively.
The unknown functions $m_i(x,t)$ and $t_i(x,t)$ satisfy the equations:

$$\Sigma_i\, m_i(x,t) = 1\ ,\ i = 1,N \tag{27a}$$

$$\Sigma_i\, t_i(x,t) = 1\ ,\ i = 1,N \tag{27b}$$

Equations (26) and (27) form a system of N+2 independent equations.
The system of equations contains 2N+4 unknowns, $m_i(x,t)$ and $t_i(x,t)$ for i = 1 to N, a(x,t), $(q_s(x,t))_{vol}$, z(x,t) and $\lambda(x,t)$ and , so, to allow its solution, some further assumptions have to be made.

4.2 Assumptions and consequences

Through all the work three common assumptions are made.

Two of them are:

- the porosity of the bed, λ, is constant in time and space, that is, $\lambda(x,t) = \lambda_0$ and
- the control volume has a constant width, that is $B(x)$ = constant.

With these assumptions equations (25) and (26) become:

$$\frac{\partial z}{\partial t} = -\frac{1}{1-\lambda_0}\frac{\partial((q_s)_{vol})}{\partial x} \text{ and} \tag{28}$$

$$\frac{\partial am_i}{\partial t} + \frac{1}{1-\lambda_0}\frac{\partial(t_i(q_s)_{vol})}{\partial x} + p_i\frac{\partial(z-a)}{\partial t} = 0\ . \tag{29}$$

Equations (29) correspond to the previously presented equations (5).

The third assumption consists in expressing the transport rate of each sediment class, $t_i(q_s)_{vol}$, as the product of the value of the fraction in the mixing layer, m_i, and its volume transport rate capacity, $(q_{si})_{vol}$:

$$t_i(q_s)_{vol} = m_i\ (q_{si})_{vol} \tag{30}$$

This leads to the equation:

$$\frac{\partial am_i}{\partial t} + \frac{1}{(1-\lambda_0)}\frac{\partial(m_i(q_{si})_{vol})}{\partial x} + p_i\frac{\partial(z-a)}{\partial t} = 0\ . \tag{31}$$

The volume transport rate capacities, $(q_{si})_{vol}$, in the general case, are assumed to depend, through known hiding functions, on the composition of the mixing layer, that is, they are known functions of the variables $m_i(x,t)$.
Equations (31) with the constraint:

$$\Sigma_i\ m_i(x,t) = 1\ ,i = 1,N \tag{27a}$$

or with equation:

$$\frac{\partial z}{\partial t} = -\frac{1}{1-\lambda_0}\frac{\partial((q_s)_{vol})}{\partial x} \tag{28}$$

constitute the basis of the model.
This system, however, is still undetermined as it contains N+2 unknown variables, $m_i(x,t)$ for i = 1 to N, $a(x,t)$ and $z(x,t)$, but only N+1 equations.
To allow its solution, a further equation or assumption must be introduced.

The decision was taken to analyse the behaviour of the model with three different assumptions to provide the extra constraint. They are referred to as SCHEME 1, SCHEME 2 and SCHEME 3.

SCHEME 1 is the simplest one.
It is obtained by imposing the following conditions along the flume and during the run:

- constant volume transport rate capacity for each sediment fraction, that is:

$$(q_{si})_{vol}(x,t) = (q_{si})_{vol} \tag{32}$$

- constant thickness of the mixing layer, that is:

$$a(x,t) = a_0. \tag{33}$$

Imposing these conditions equations (31) become:

$$a_0 \frac{\partial m_i}{\partial t} + \frac{1}{(1-\lambda_0)} \frac{\partial (m_i (q_{si})_{vol})}{\partial x} + p_i \frac{\partial z}{\partial t} = 0 . \tag{34}$$

With the assumption that the third term is negligible in comparison with the second one, provided that this one is different from zero, equations (34) for the movable and non-movable sediment fractions, respectively, take the form:

$$\frac{\partial m_i}{\partial t} + \frac{(q_{si})_{vol}}{a_0(1-\lambda_0)} \frac{\partial m_i}{\partial x} = 0 \ , \ i = 1,M \ , \tag{35a}$$

$$a_0 \frac{\partial m_i}{\partial t} + p_i \frac{\partial z}{\partial t} = 0 \ , \ i = M+1,N \ . \tag{35b}$$

The general solution to SCHEME 1 can be obtained from the system that these equations form with the constraint:

$$\Sigma_i \ m_i(x,t) = 1 \ , i = 1,N \ . \tag{27a}$$

SCHEME 2 is obtained by assuming that the thickness of the mixing layer is constant in time and space, that is, introducing the condition:

$$a(x,t) = a_0 \ , \tag{33}$$

in equations:

$$\frac{\partial a m_i}{\partial t} + \frac{1}{(1-\lambda_0)} \frac{\partial (m_i (q_{si})_{vol})}{\partial x} + p_i \frac{\partial (z-a)}{\partial t} = 0 . \tag{31}$$

The solution to the problem comes from a system of N+1 independent equations in N+1 unknown functions, that is $m_i(x,t)$, for i = 1 to N, and $z(x,t)$:

$$a_0 \frac{\partial m_i}{\partial t} + p_i \frac{\partial z}{\partial t} = 0 \ , \ i = M+1,N \tag{36b}$$

plus the constraint (27a): $\Sigma_i \, m_i(x,t) = 1$.
The rearrangement of the equations produces an equivalent system consisting of equations (36) and an equation directly expressing the variation in time of the bed level:

$$\frac{\partial z}{\partial t} = - \frac{1}{(1-\lambda_0)} \Sigma_i \left(\frac{\partial (m_i (q_{si})_{vol})}{\partial x}\right) . \tag{37}$$

This scheme physically represents a mixing layer of constant thickness, moving downwards as erosion takes place, continuously being fed by the undisturbed underlying parent bed.
The value of the functions $m_i(x,t)$ refers to the ratio between the volume of each sediment fraction and the volume occupied by all the fraction in the active layer. The last term, because of the assumptions of constant porosity and thickness of the active layer, does not vary in time and space.

In **SCHEME 3** the term (z(x,t)-a(x,t)), representing the elevation of the bottom of the mixing layer, is treated as a constant.
This assumption leads to the equations:

$$\frac{\partial a m_i}{\partial t} + \frac{1}{(1-\lambda_0)} \frac{\partial (m_i (q_{si})_{vol})}{\partial x} = 0 \ , \ i=1,M \tag{38a}$$

$$\frac{\partial a m_i}{\partial t} = 0 \ , \ i=M+1,N \ , \tag{38b}$$

which has to be to be combined with the constraint:

$$\Sigma_i \, m_i(x,t) = 1 \ , i = 1,N \tag{27a}$$

From the rearrangement of these equations, the (N+1)-th equation can also take the form:

$$\frac{\partial a}{\partial t} = - \frac{1}{(1-\lambda_0)} \Sigma_i \left(\frac{\partial (m_i (q_{si})_{vol})}{\partial x} \right) . \tag{39}$$

It can be noticed that equations (37) and (39), obtained, respectively, by the summation of equations (36) and (38), both represent the application of the principle of continuity to the mixture as a whole.
In this scheme the derived functions $m_i(x,t)$ describe the variation of the ratio between the volume of each sediment fraction and the volume of all the sediment in an active layer of decreasing thickness a(x,t).

4.3 Analytical and numerical solutions

The particular solution to the differential equations describing the model in its three schemes, is dependent upon the initial and boundary conditions assumed, that is, on the values of the unknown variables:

$m_i(x,t)$ in SCHEME1,
$m_i(x,t)$ and $a(x,t)$ in SCHEME2,
$m_i(x,t)$ and $z(x,t)$ in SCHEME3,

for $t = 0$ along the flume (**initial conditions**) and for $x = 0$ during all the run (**boundary conditions**).

For all the schemes, the initial condition imposed on the sediment consists in the equivalence of the composition of the parent bed, p_i, and of the mixing layer.
Mathematically this is translated into the constraint:

$$m_i(x,0) = p_i \tag{40}$$

for $x = 0$ to LL, where LL is the length of the flume.

Depending on the further assumptions made, the differential equations characterising the schemes described above can be solved either analytically or numerically.
In this work the equations have been solved numerically using a backward finite difference scheme.

SCHEME 1

Equations (35) can be solved analytically.
Referring to the movable sediment fractions, equations (35a) form a set of linear differential equations of the first order and first degree, that can be solved separately from the remaining equations.
They represent M independent linear waves, describing the motion, in the positive x direction, of a perturbation to the initial conditions defined by the boundary conditions.
The boundary conditions at the upstream end of the flume initiate the perturbations that move downstream with a celerity given by:

$$c_i = \frac{(q_{si})_{vol}}{a_0(1-\lambda_0)} \;,\; i=1,M \;. \tag{41}$$

The particular solution of the problem is obtained by adding to the initial conditions (40) the boundary conditions:

$$m_i(0,t) = 0 \;,\; i = 1, M \;, \tag{42a}$$

$$m_i(0,t) = p_i \;,\; i = M+1, N \;, \tag{42b}$$

respectively referring to movable and non movable fractions and simulating the absence of sediment inflow at the beginning of the flume during all the run.

To match these conditions, the particular solution to equations (35a) becomes:

$$m_i(x,t) = p_i \quad \text{if} \quad t < x/c_i$$
$$m_i(x,t) = 0 \quad \text{if} \quad t >= x/c_i \ , \quad i = 1,M \ . \qquad (43a)$$

A fuller explanation of this result is presented in Appendix C.
For the non movable sediment fractions, that is for i = M+1 to N, the problem is simplified by observing that equations (35b), which state that the total amount of sediment along the flume does not vary in time, can be written as:

$$a_0 \ m_i(x,t) + p_i \ z(x,t) = \text{Const} = a_0 \ p_i + p_i \ z(x,0). \qquad (35b')$$

Summing all these equations from i = M+1 to N and substituting the obtained expression of z(x,t) in each of them, the variation in time and space of the i-th fraction is a function of the initial distribution and of the actual distribution of the movable fractions:

$$m_i(x,t) = p_i \frac{\sum_{i=M+1,N} m_i \ (x,t)}{\sum_{i=M+1,N} p_i} = p_i \frac{1-\sum_{i=1,M} m_k(x,t)}{\sum_{i=M+1,N} p_i} \ . \qquad (43b)$$

Physically, the relations (43a) indicate that at the end of the flume an initial interval of constant transport rate is followed, for $t > LL/(c_i)_{max}$, by a gradual decrease in transport rate.
After an interval of time equal to LL/c_i has elapsed, no further material of the i-th sediment fraction arrives at the end of the flume. From that time onwards, the value of $m_i(x,t)$ is equal to zero along the entire length of the flume.
As the celerity of the waves is directly proportional to the transport rate capacities of the sediment fractions, the transport of the finest particles will stop before that of the coarsest ones.
When $t > LL/(c_i)_{min}$, the static armour condition is obtained.

SCHEME 2
The general formulation of the problem, as by equations (36) and (37), does not admit an analytical solution and a numerical approximation has been used.
The numerical solution is obtained using a finite difference grid with a time step Δt and a space step Δx.
Equations (36a), referring to the movable sediment fractions, that is, to i = 1 to M, with the use of equation (37) are approximated by:

$$\frac{m_i \ (x,t+\Delta t) \ - \ m_i \ (x,t)}{\Delta t} \ a_0 \ (1-\lambda_0) \ =$$

$$= - \frac{(m_i(x,t)(q_{si})_{vol}(x,t) \ - \ m_i(x-\Delta x,t)(q_{si})_{vol}(x-\Delta x,t))}{\Delta x} \ +$$

$$+ \ p_i \frac{\sum_i (m_i(x,t)(q_{si})_{vol}(x,t) \ - \ m_i(x-\Delta x,t)(q_{si})_{vol}(x-\Delta x,t))}{\Delta x} \ . \qquad (44a)$$

It can be observed that equations (36b), relative to the non movable fractions, coincide with equations (35b) of SCHEME 1.
Therefore their solution, not requiring a finite difference approximation, is:

$$m_i(x,t) = p_i \frac{\sum_{i=M+1,N} m_i(x,t)}{\sum_{i=M+1,N} p_i} = p_i \frac{1-\sum_{i=1,M} m_i(x,t)}{\sum_{i=M+1,N} p_i} . \tag{44b}$$

The particular solution of the problem, as expressed by equations (44), is obtained by adding to the initial conditions (40) the relation:

$$z(x,0) = z_0 \tag{45}$$

and the boundary conditions:

$$m_i(0,t) = 0 , \qquad i = 1,M , \tag{46a}$$

$$m_i(0,t) = p_i/\Sigma_i p_i , \qquad i = M+1,N , \tag{46b}$$

referring to movable and non movable fractions, respectively.
These equations simulate the absence of sediment inflow at the upstream end of the flume during all the run.

In the particular case of assuming constant transport rate capacities of all the sediment fractions, the problem can be described by the equations:

$$a_0 \frac{\partial m_i}{\partial t} + \frac{(q_{si})_{vol}}{(1-\lambda_0)} \frac{\partial(m_i)}{\partial x} + p_i \frac{\partial z}{\partial t} = 0 , \ i=1,M , \tag{47a}$$

$$a_0 \frac{\partial m_i}{\partial t} + p_i \frac{\partial z}{\partial t} = 0 , \ i=M+1,N , \tag{47b}$$

$$\frac{\partial z}{\partial t} = - \frac{1}{(1-\lambda_0)} \sum_i \left((q_{si})_{vol} \frac{\partial(m_i)}{\partial x} \right) . \tag{48}$$

This system was analytically solved by Gilbert (1992), as described in Appendix D, with the initial conditions (40) and the boundary conditions (45).

As in SCHEME 1, the final condition of absence of transported material on an armoured surface is reached after the arrival at the end of the flume of the M-th perturbation wave, related to the M movable sediment fractions.
In this case the calculation of the wave celerity requires some mathematical manipulation.

SCHEME 3
This scheme also does not allow an analytical solution.
To perform a numerical approximation equations (38a) and (39) are written introducing, again, a finite difference grid with timestep Δt and space step Δx.

As in SCHEME 1, the final condition of absence of transported material on an armoured surface is reached after the arrival at the end of the flume of the M-th perturbation wave, related to the M movable sediment fractions.
In this case the calculation of the wave celerity requires some mathematical manipulation.

SCHEME 3
This scheme also does not allow an analytical solution.
To perform a numerical approximation equations (38a) and (39) are written introducing, again, a finite difference grid with timestep Δt and space step Δx.

The equations become:

$$\frac{m_i(x,t+\Delta t) - m_i(x,t)}{\Delta t} a(x,t) (1-\lambda_0) = \tag{48a}$$

$$= - \frac{(m_i(x,t)(q_{si})_{vol}(x,t) - m_i(x-\Delta x,t)(q_{si})_{vol}(x-\Delta x,t))}{\Delta x} +$$

$$+ m_i(x,t) \frac{\sum_i (m_i(x,t)(q_{si})_{vol}(x,t) - m_i(x-\Delta x,t)(q_{si})_{vol}(x-\Delta x,t))}{\Delta x}$$

and

$$\frac{a(x,t+\Delta t) - a(x,t)}{\Delta t} (1-\lambda_0) =$$

$$= \frac{\sum_i (m_i(x,t) (q_{si})_{vol}(x,t) - m_i(x-\Delta x,t)(q_{si})_{vol}(x-\Delta x,t))}{\Delta x},$$

with i = 1 to M.

Equations (38b), for the non movable sediment fractions, can be directly solved as a function of the movable fractions using the equations:

$$m_i(x,t) = \frac{a(x,0)p_i}{a(x,t)}, \quad i=M+1,N \tag{48b}$$

The particular solution of the problem, as expressed by equations (48) and (49), is obtained by adding to the initial conditions (40) the relation:

$$a(x,0) = a_0 \tag{50}$$

and the boundary conditions:

5 Test of the model

The model previously described was tested using experimental data from Proffitt (1980) and Tait et al. (1992).
Two aspects of the static armour formation were analysed: the variation in time of the total sediment transport rate and the final composition of the armoured surface.

5.1 Total transport rate in time

As a first step, the run 1-2 by Proffitt (1980) was simulated.
This run was chosen to study, from the six runs that he performed with the same initial sediment (mixture 1).
As the individual sediment transport measurements at the downstream end of the flume were not available, the curve fitted by Proffitt to the data was used.
This curve consisted of two lines in a log-log plane: the first one reproduced the initial phase of constant transport rate, while the second part of the curve showed a decreasing sediment transport rate.
The distribution of the parent bed and of the armoured surface and the value of the hydraulic and geometric quantities involved are presented in Appendix E.
Through all the present work a value of 30% was adopted for the porosity λ_0.

SCHEME 1 - The value for the constant thickness of the mixing layer, a_0, was assumed equal to D_{100}, that is, equal to the maximum size of the parent bed material.
The analytical solution, obtained by using Meyer-Peter & Müller (1948) and Ackers & White (1973) sediment transport formulas, described in Appendices A and B, is presented in figure 3.
Even if the initial transport rate differs from the measured one, this simple scheme provides an unexpected good agreement in the definition of the slope of the function describing the variation of the total sediment transport in time.

It was noticed that the slope of the solution depends not only on the sediment transport formula adopted, but also on the discretisation performed to describe the mixture.
Results presented in figure 3 were obtained schematizing the mixture using the sieve data.
A polynomial interpolation, based on splines, was used to increase the number of size classes in the model with the aim of more closely approximating the continuous sediment distribution curve.
More details on the spline theory can be found in Mizumura (1985).
The results obtained using the Meyer-Peter & Müller formula with the sieving data and with the "spline distribution" are compared in Figure 4.

As a consequence of the linear dependence of each perturbation celerity, expressed by equation (41), to the product $a_0(1-\lambda_0)$, variations in the porosity of the bed and in the thickness of the mixing layer produce a translation of the solution parallel to the time axis.
Results obtained with both Meyer-Peter & Müller and Ackers & White formulae, adopting a thickness of the mixing layer equal to D_{50} and a porosity equal to 20%, are presented in figure 5.
It can be noticed that, for this run, a better agreement was obtained with a mixing layer equal to D_{100} and a porosity of 30%: even if the slope of the curve does not change, with a thickness equal D_{50} and a porosity of 20%, the

total eroded material, visualized in the diagram by the area beneath the curve, is very far from that observed by Proffitt.

SCHEME 2 - For the mixing layer a thickness equal to D_{100} was assumed also in this scheme.
Initially no hiding factors were introduced.
The analytical solution obtained by using Meyer-Peter & Müller and Ackers & White formulae is shown in Figure 6.
By comparing the results of SCHEME 1 with SCHEME 2, it would appear that no advantage seems to be gained by the adoption of the more complex SCHEME 2.
The analytical solution and results of the numerical calculation carried on with a ratio $\Delta x/\Delta t = 1$, by using Meyer-Peter & Müller formula is presented in Figure 7.
This comparison was performed in order to get an indication of the value of the ratio $\Delta x/\Delta t$ to be adopted, in the succeeding numerical calculations, in order to ensure convergence to the solution.
As no substantial improvement was noticed by using a ratio $\Delta x/\Delta t < 1$, this value was adopted through all the work.
It can be seen that the values following the inflection point in the numerical solution do not have a real correlation; these values are merely a product of the numerical approximation when reproducing the sudden absence of sediment transport.

No analytical solution is available for this scheme if hiding functions are introduced.
In figure 8 results are presented of the use of Meyer-Peter & Müller formula with Egiazaroff hiding function and Ashida & Michiue modification.
The introduction of the hiding function produces, as expected, a lower value of the initial sediment transport rate, but no improvement in the simulation of the slope of the experimental data is obtained.

SCHEME 3 - Initially Meyer-Peter & Müller and Ackers & White formulas were applied, with no hiding correction, with the initial thickness of the mixing layer assumed to be equal to D_{100}.
The numerical solution obtained with a ratio $\Delta x/\Delta t = 1$, as adopted in the previous scheme, obtained with Meyer-Peter & Müller and Ackers & White formulas is presented in Figure 9.
A very small improvement seems to come from the use of this scheme.
As noticed for SCHEME2, the values of the numerical solutions following the inflection points may have been generated by the numerical approximation, that is they may not have a real correlation.

In order to consider the interactions between the fractions composing the sediment mixture, the Egiazaroff (1965) hiding function, corrected by Ashida & Michiue (1971), was coupled with Meyer-Peter & Müller sediment transport formula and Ackers & White (1973) sediment transport formula was associated with the Day (1980) correction.
Results are presented in Figure 10.
With reference to Meyer-Peter & Müller formula, the slope of the solution obtained with hiding correction seems closer to the slope defined by Proffitt's data, but the solution underestimates the total eroded material.
Uncertain results come from the Ackers & White formula and further investigations should be carried out to give a meaning to the final trend of constant transport rate.

It should be checked if this trend is reproducing a real behaviour of the model, depending on the formulation of the hiding function, or if it has just a numerical explanation.

A simple attempt was made to modify the expression of the hiding function applied to Meyer-Peter & Müller sediment transport formula.
The influence of the hiding function on part of the fraction was tested.
Figure 11 shows a comparison between the results obtained using a hiding correction for all the fractions of the mixture and for the coarsest ones.
The coarsest sizes were defined using the geometric mean diameter of the mixture, D_g, calculated step by step along the flume; their characteristic dimension verifies the relation: $D_i > D_g$. The hiding correction applied only to the coarsest fractions produces, as expected, a better definition of the initial transport rate, but a very small improvement is achieved in the fitting of the slope of the experimental curve.

The second step was to analyse run 4 performed by Tait et al. (1992).
The distribution of the parent bed and the value of the hydraulic and geometric quantities involved are presented in Appendix F.
Sediment transport formulae by Meyer-Peter & Müller (1948) and Ackers & White (1973) were introduced in SCHEME 1, adopting a mixing layer of fixed thickness equal to D_{100}.
Results are presented in Figure 12.
The Meyer-Peter & Müller formula gives a better fit to the experimental data, even if the initial value of the transport rate is overestimated.
Ackers & White sediment transport formula, on the contrary, underestimates the initial value of the transport rate.
It can be seen that both of the solutions present a slope very close to the experimental one and it would be of interest to check if this occurrence is present in further applications of the scheme to different runs.

5.2 Sediment distribution at the surface

The composition of the armour layer obtained experimentally with the one produced with the different schemes of the model was compared.

If no hiding functions are introduced, SCHEME 1, SCHEME 2 and SCHEME 3 give the same final distribution of the mixing layer.
This layer results to be composed by those fractions of the mixture that would not be movable in a uniform sediment of their size.
In the first two schemes the calculated percentages are referred to the fixed thickness of the mixing layer, while, in the third one, they are relative to the final value of it.
Referring to Proffitt run 1-2, Figure 13 shows the grading curves of the sampled armour layer and of the layers obtained by using both Meyer-Peter & Müller and Ackers & White sediment transport formulae.
Good agreement is obtained for the coarsest fractions with Meyer-Peter & Müller sediment transport formula.
The discrepancies relating to the finer fractions are due to the fact that the model, if no hiding functions are introduced, gives a final armour layer only composed of the non movable fractions.

Figure 14 shows the final distribution obtained by using Meyer-Peter & Müller (1948) sediment transport formula and Egiazaroff (1965) hiding function, modified by Ashida & Michiue (1971). The agreement with the experimental data is good.

Finally, the effect was analysed of a modification in the hiding function, when using the Meyer-Peter & Müller sediment transport formula.
Egiazaroff - Ashida & Michiue hiding function was applied to all the fractions of the mixture and to the finest one, respectively.
Results are presented in Figure 15.
In both cases the agreement with experimental data seems to be good and no evident differences come from the modifications introduced.
Further investigations should be carried out to define if there is any influence of the boundary conditions, imposing the armoured distribution at the upstream end of the flume, on the results.

6 Implications for river mechanics

Though the experiments considered represent a very idealised system and though the models developed are simplistic in form, the results obtained have implications for river mechanics. The fact that the model that has been developed can provide a reasonable description of the system suggests that the development of armouring in a river can be at least approximately modelled by a similar approach using simple kinematic waves. The approach should be applicable both to the prediction of static armour but also to the development of armour under mobile conditions.

A notable feature of the simulations of the laboratory experiments was the wide disparity of propagation speeds for the different sediment sizes. This was for a laboratory experiment with a relatively limited range of grain sizes. For natural sediments which commonly have a much wider range of sizes, the disparity in the propagation speeds will be even larger. This must be considered when interpreting observations within real river systems. An appreciation of both the propagation speeds and the time since an event occurred is required before we can correctly assess the downstream impact of events.

7 Conclusions and Recommendations

7.1 Conclusions

A simple model based on the propagation of kinematic waves has been shown to predict the development of self-armoured beds. The model treats different sediment sizes independently and assumes that information on sediment supply propagates downstream at a rate dependent upon the sediment size.

It has been shown that for self-armouring experiments the model reproduces the characteristic shape of the plot of sediment transport rate with time. This curve shows an initially uniform transport rate followed after a period of time by a steadily reducing transport rate. The model indicates that this arises due to the different speeds with which information on the upstream boundary condition propagate down the flume.

A one-layer model was used to reproduce the variations of the sediment transport rate and the final composition of the surface, under static armour conditions.
Three schemes were developed, based on different assumptions regarding the thickness of the mixing layer.
The model was tested against experimental data.

SCHEME 1 gives the best results compared with transport rate data, despite its simplicity.
In this scheme a spline function was introduced to shape the grading curve of the mixtures.
Further investigations should be carried out to test the effectiveness of this representation of the mixtures in both SCHEME 2 and SCHEME 3.
Meyer-Peter & Müller (1948) and Ackers & White (1973) sediment transport formulae were used in the three schemes.
The introduction of the hiding functions of Egiazaroff (1965), Ashida & Michiue (1971) and Day (1980) in SCHEME2 and SCHEME3 did not produce the expected improvement in the fitting of experimental data.
Different sediment transport formulae and hiding functions should be adopted to validate the obtained results.
A good agreement in the final surface composition was obtained with SCHEME2 and SCHEME3; the influence of the upstream boundary conditions on the results should be verified.

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Figures

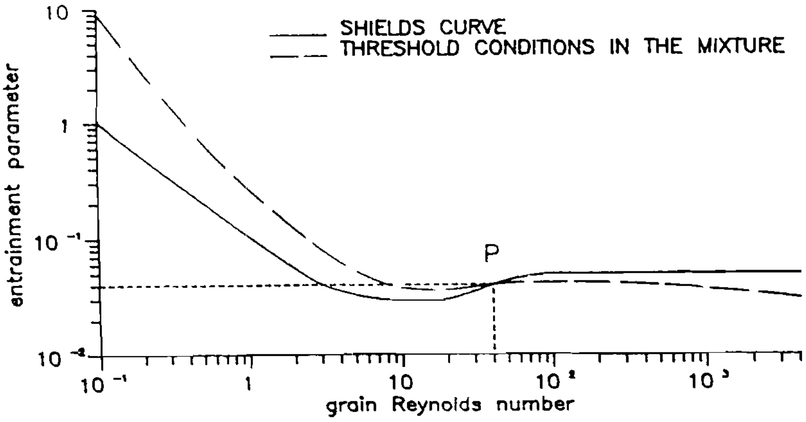

Figure 1 Shields curve for non-uniform sediments

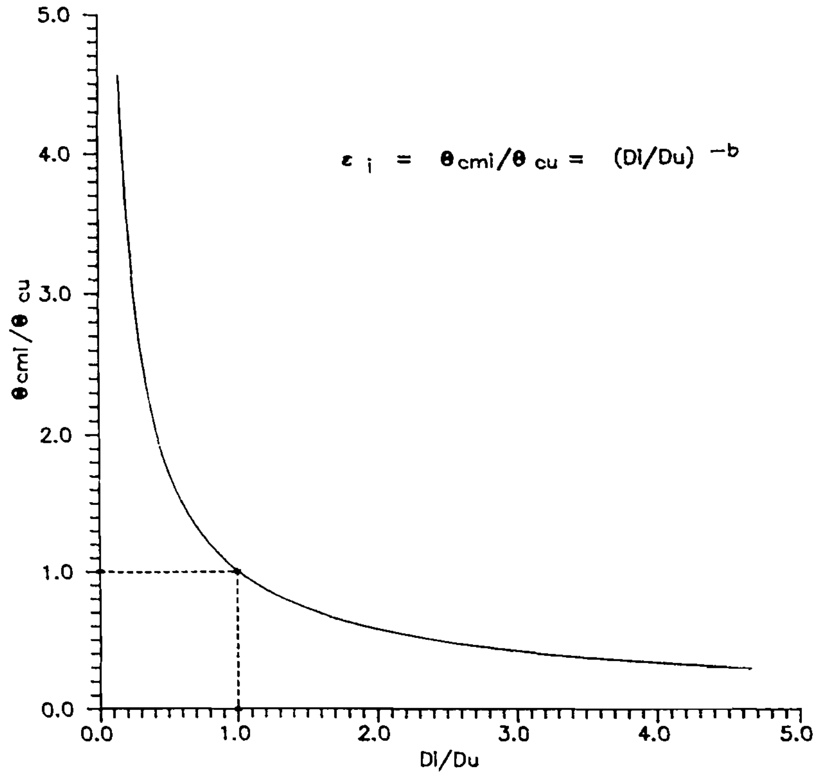

Figure 2 Hiding function ε_i as function of sediment size (eq. (13))

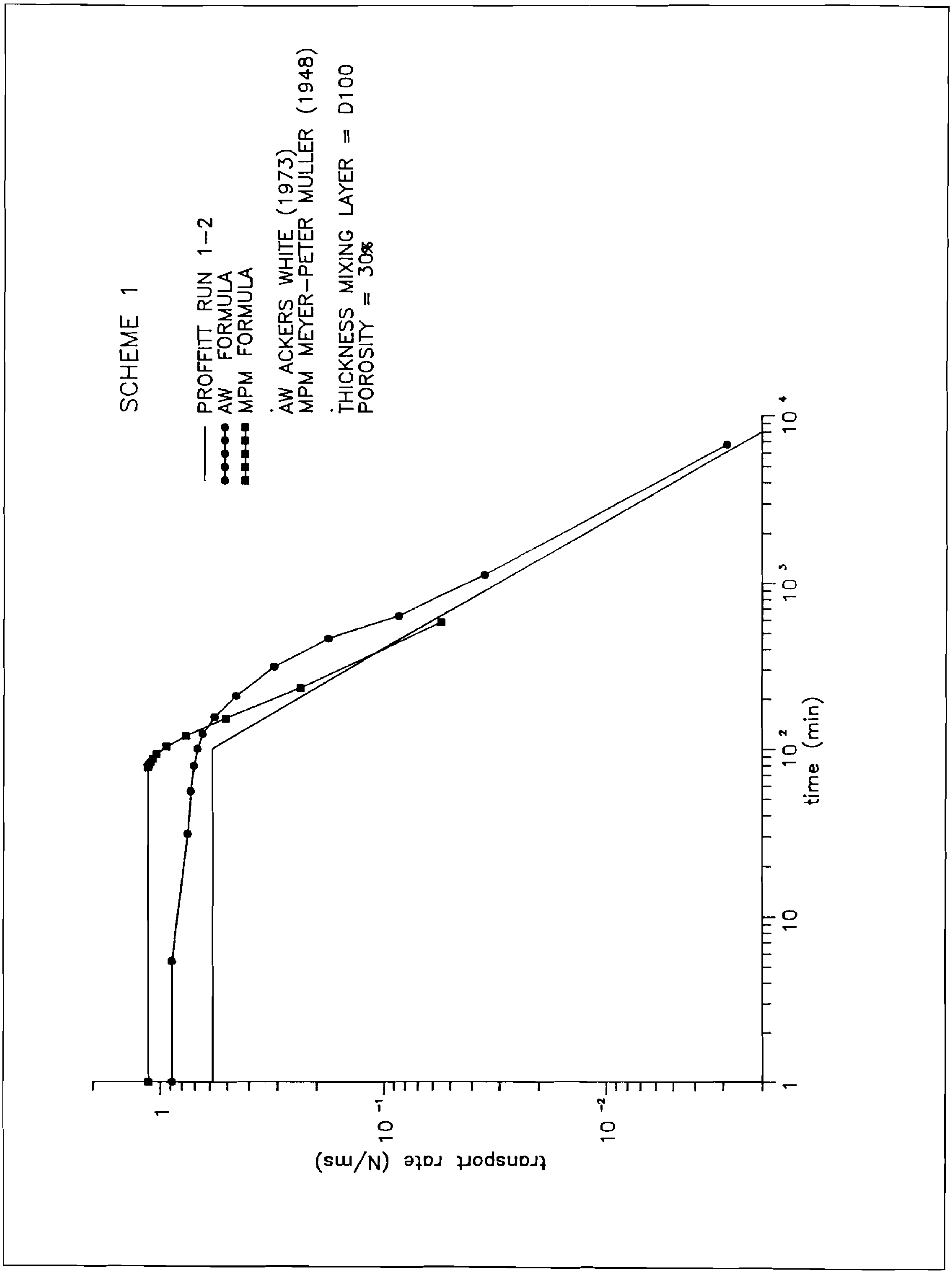

Figure 3 Scheme 1: Analytical solution, Meyer-Peter & Müller and Ackers & White sediment transport formulae

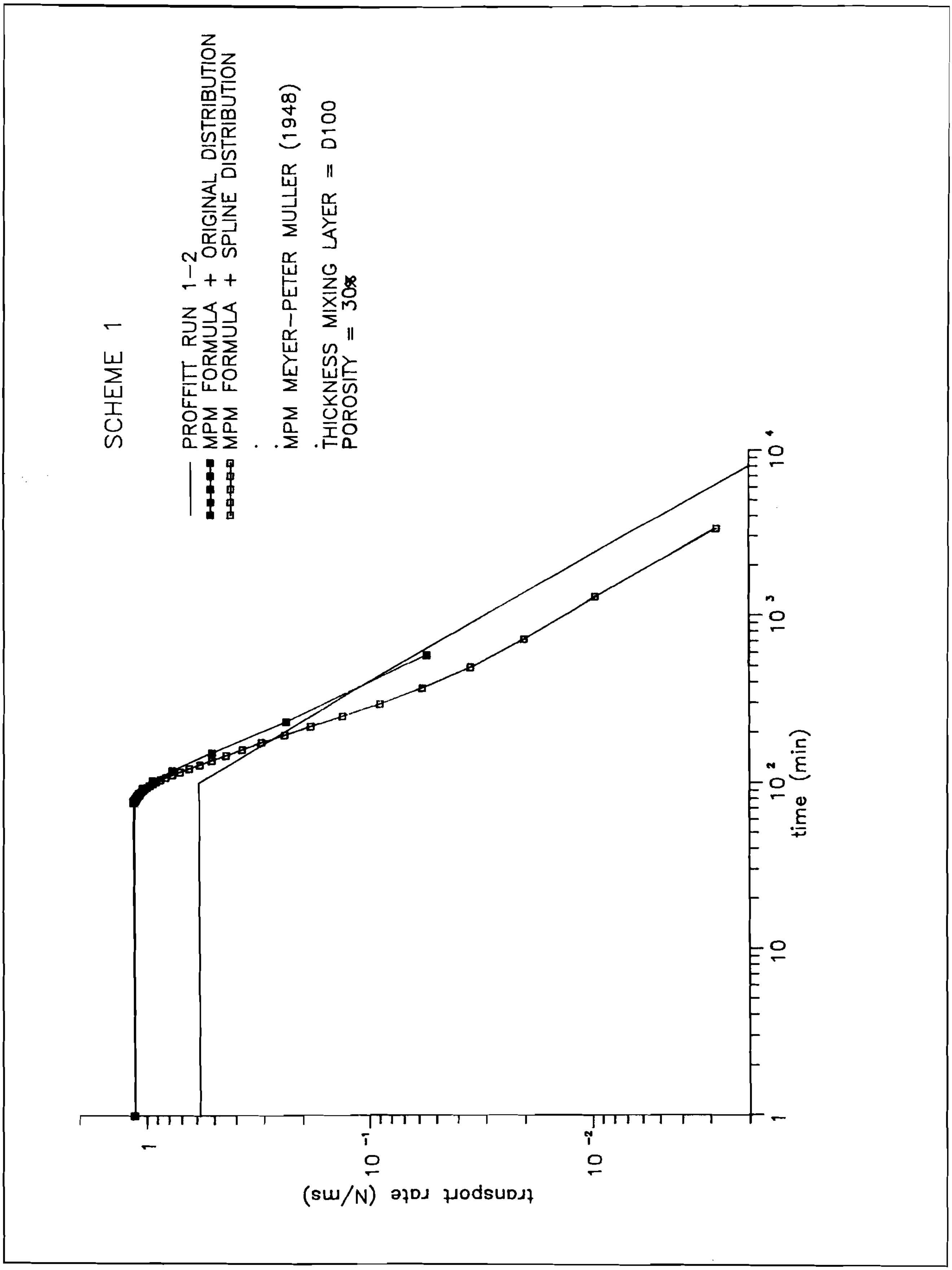

Figure 4 Scheme 1: Analytic solution, Meyer-Peter & Müller sediment transport formula, original and spline sediment distributions

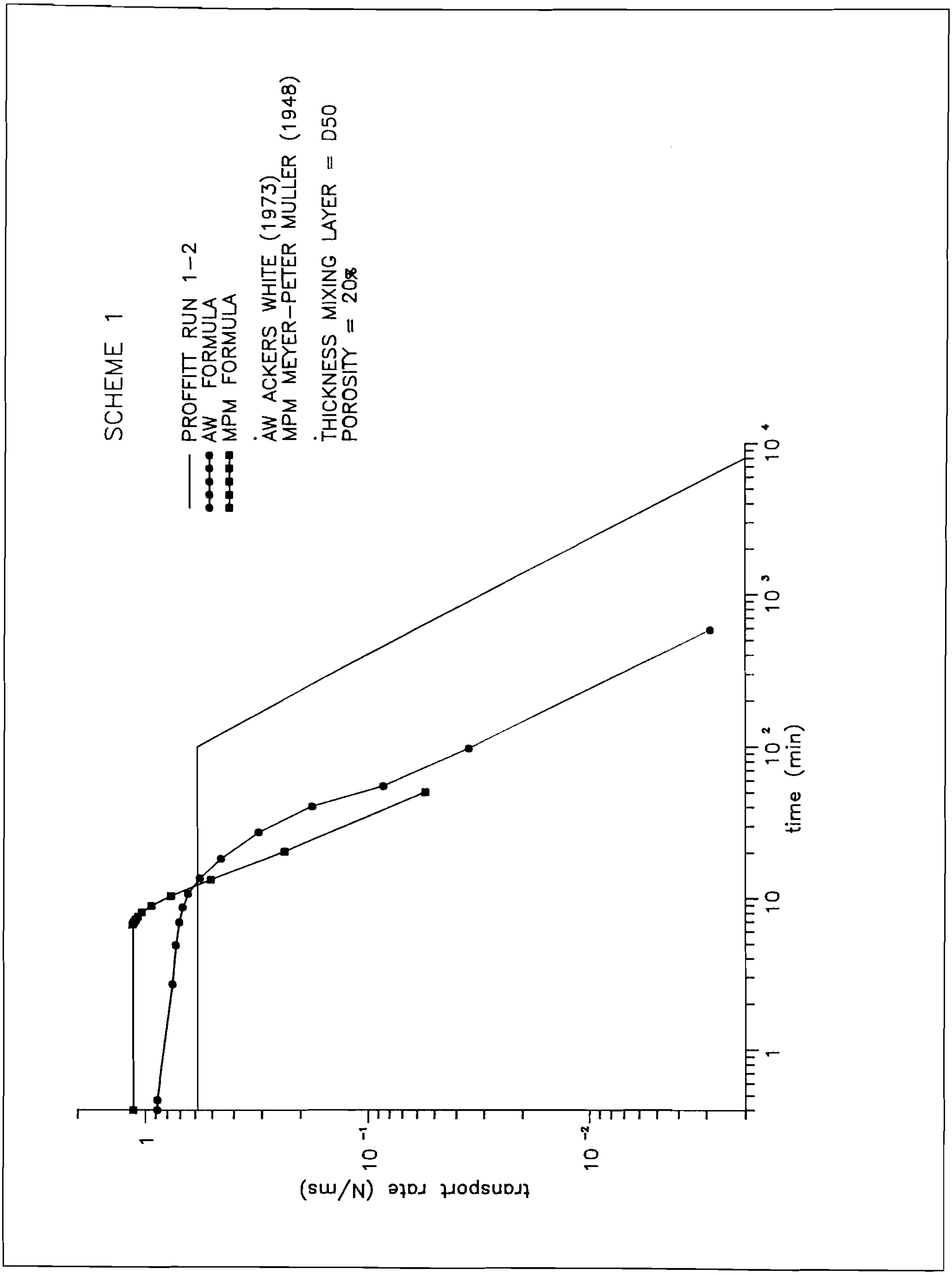

Figure 5 Scheme 1: Analytic solution, Meyer-Peter & Müller sediment transport formula, modified porosity and thickness of mixing layer

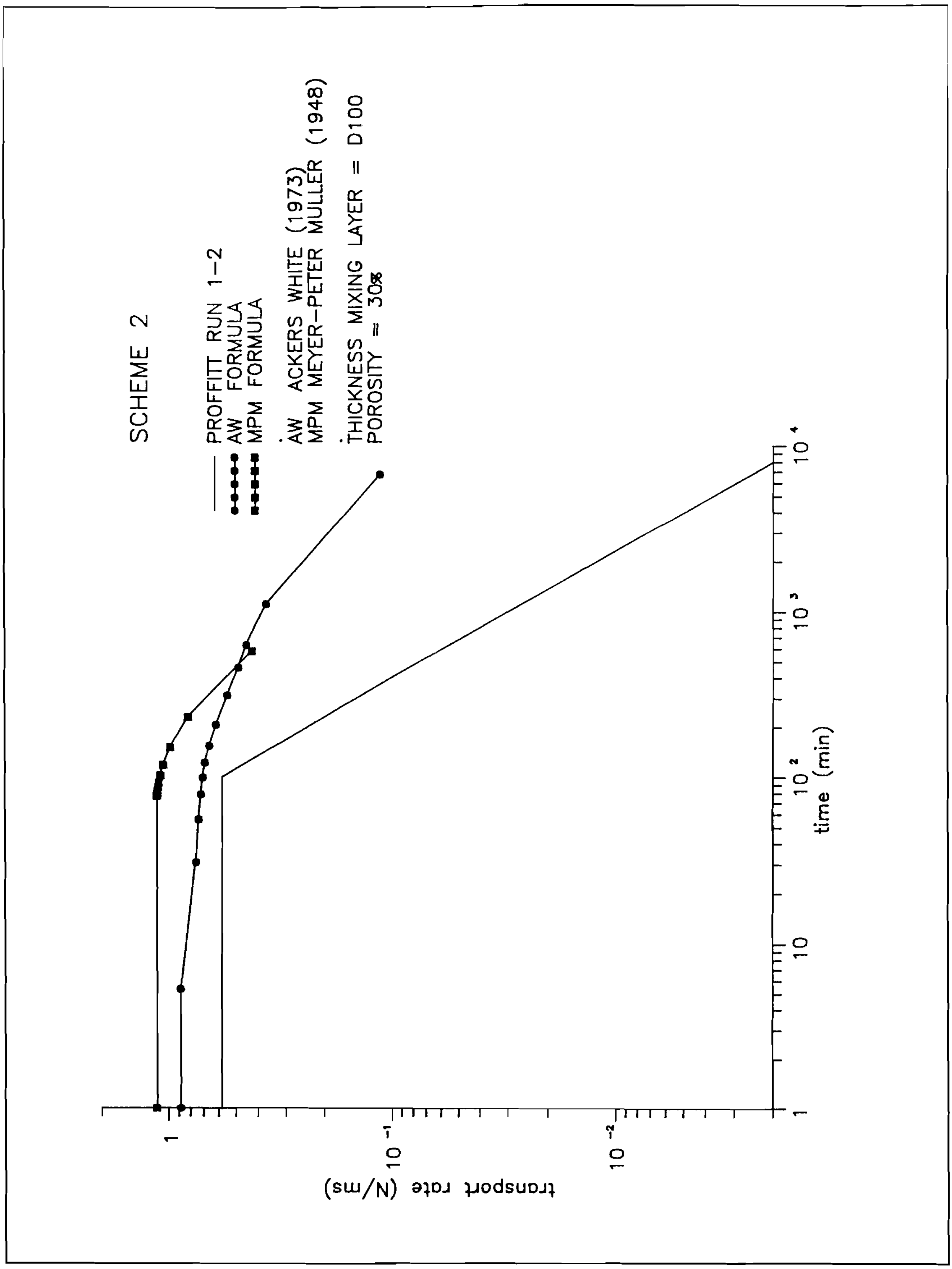

Figure 6 Scheme 2: Analytic solution, Meyer-Peter & Müller and Ackers & White sediment transport formulae

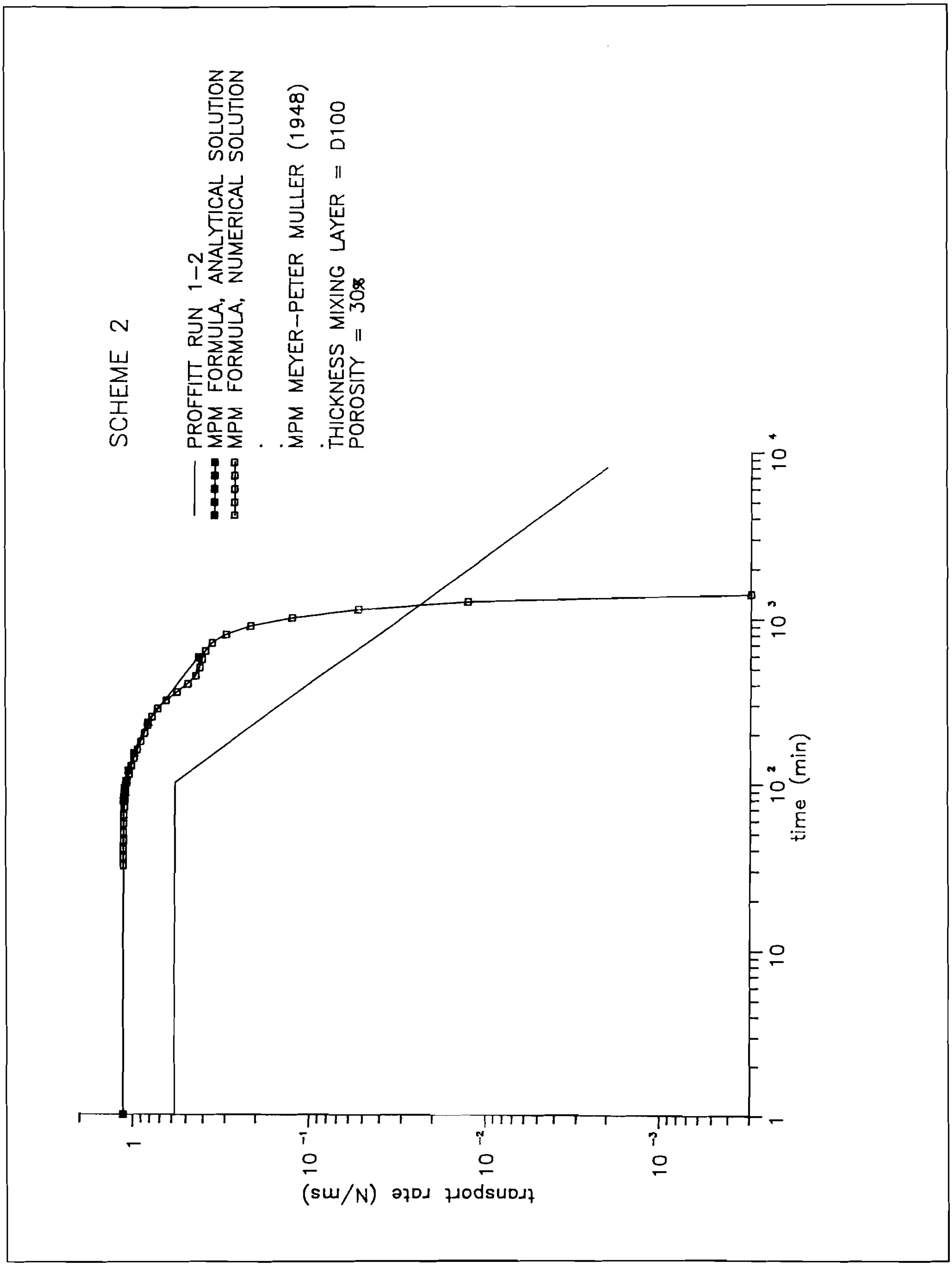

Figure 7 **Scheme 2: Analytic and numerical solutions, Meyer-Peter & Müller sediment transport formula**

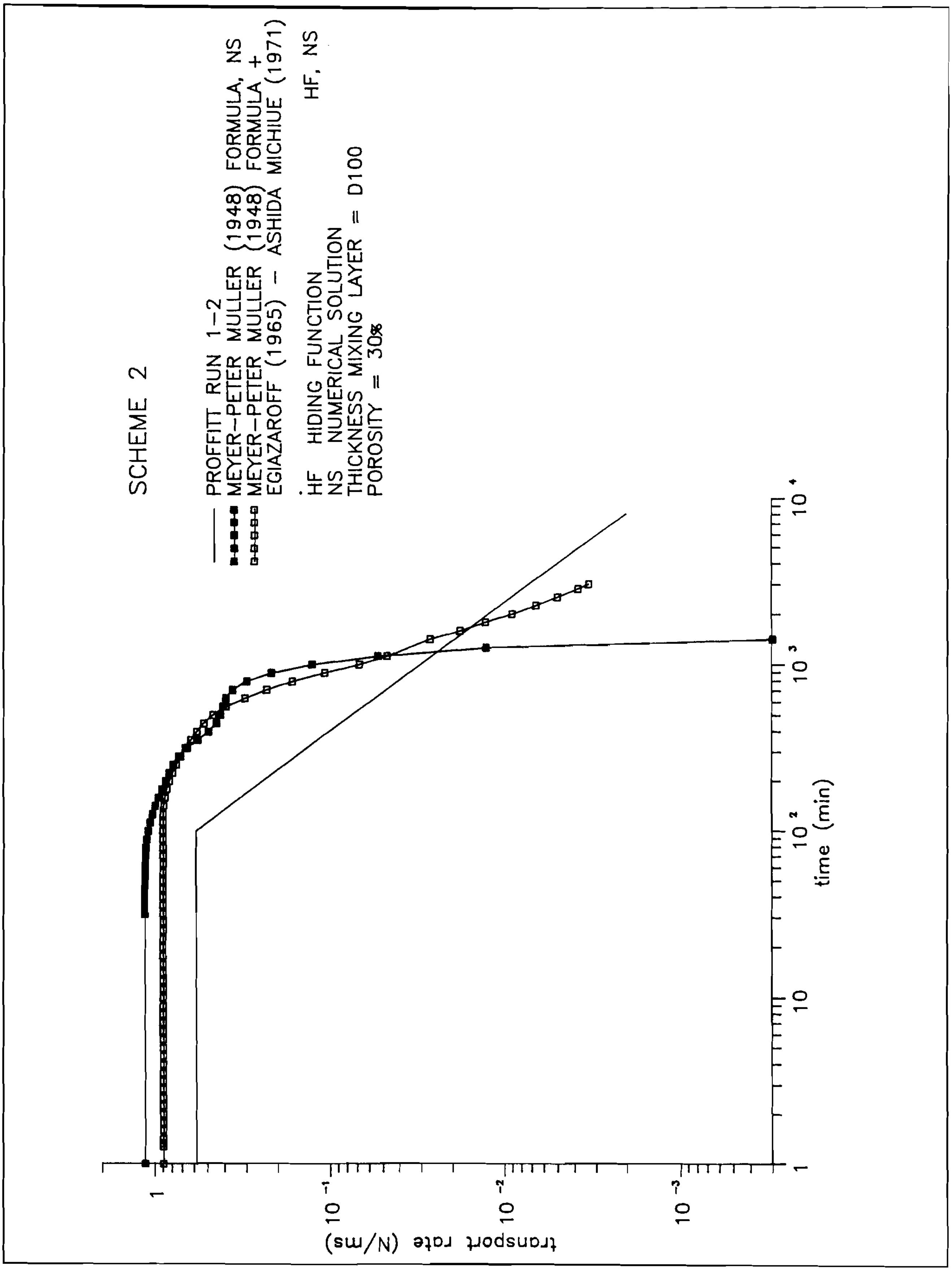

Figure 8 Scheme 2: Numerical solution, Meyer-Peter & Müller sediment transport formula, with hiding functions

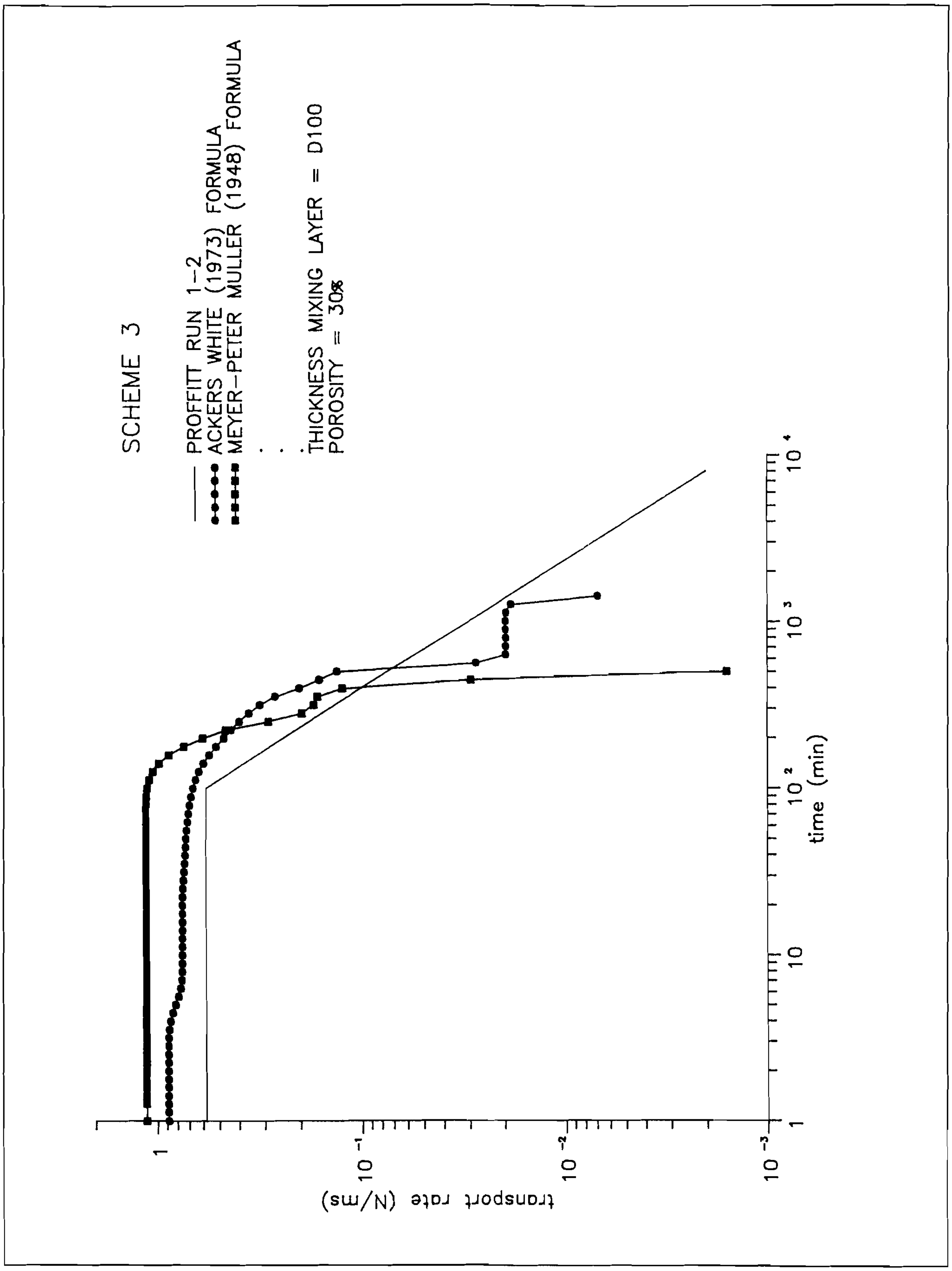

Figure 9 Scheme 3: Numerical solutions

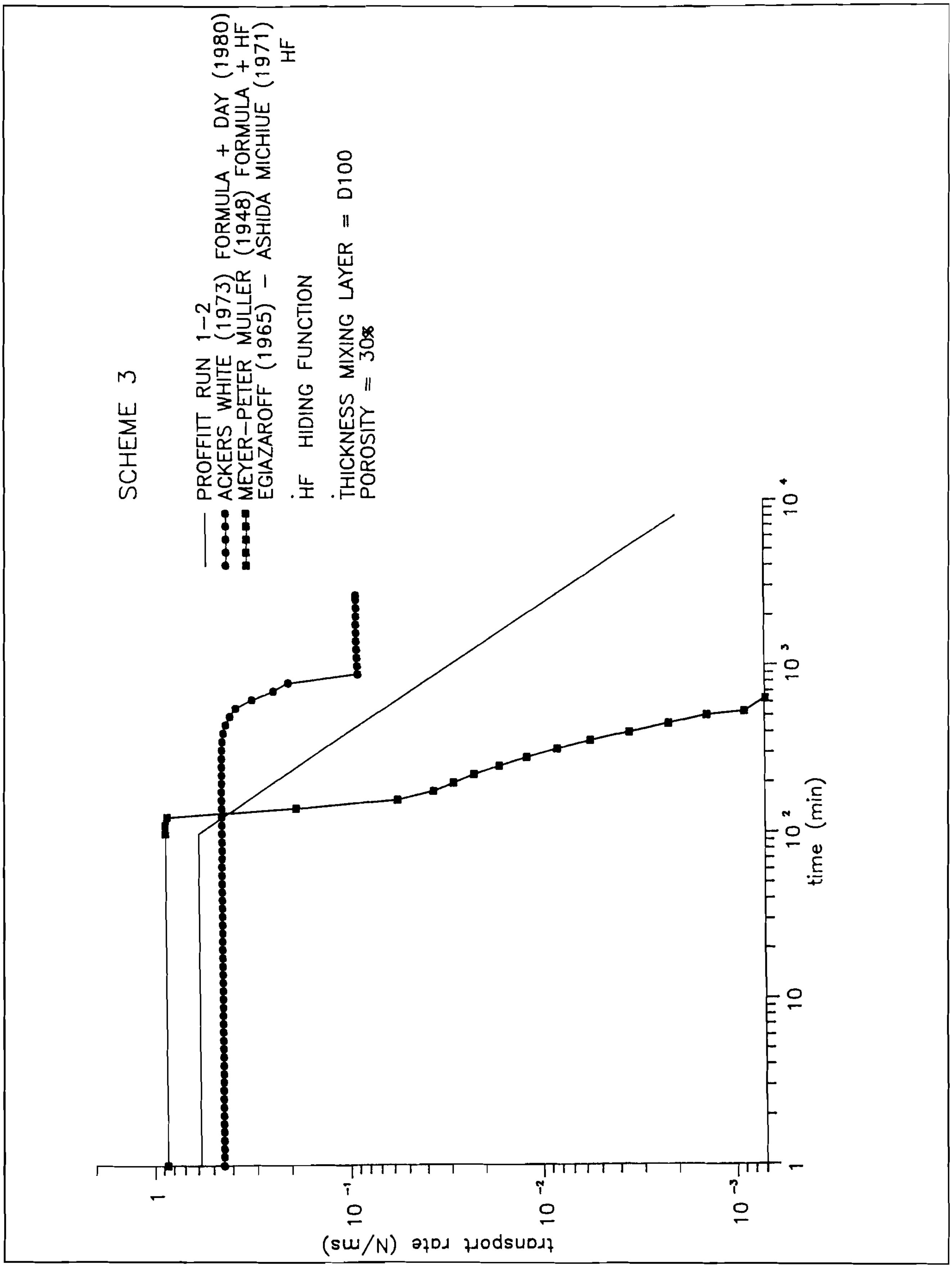

Figure 10 Scheme 3: Numerical solutions with hiding function

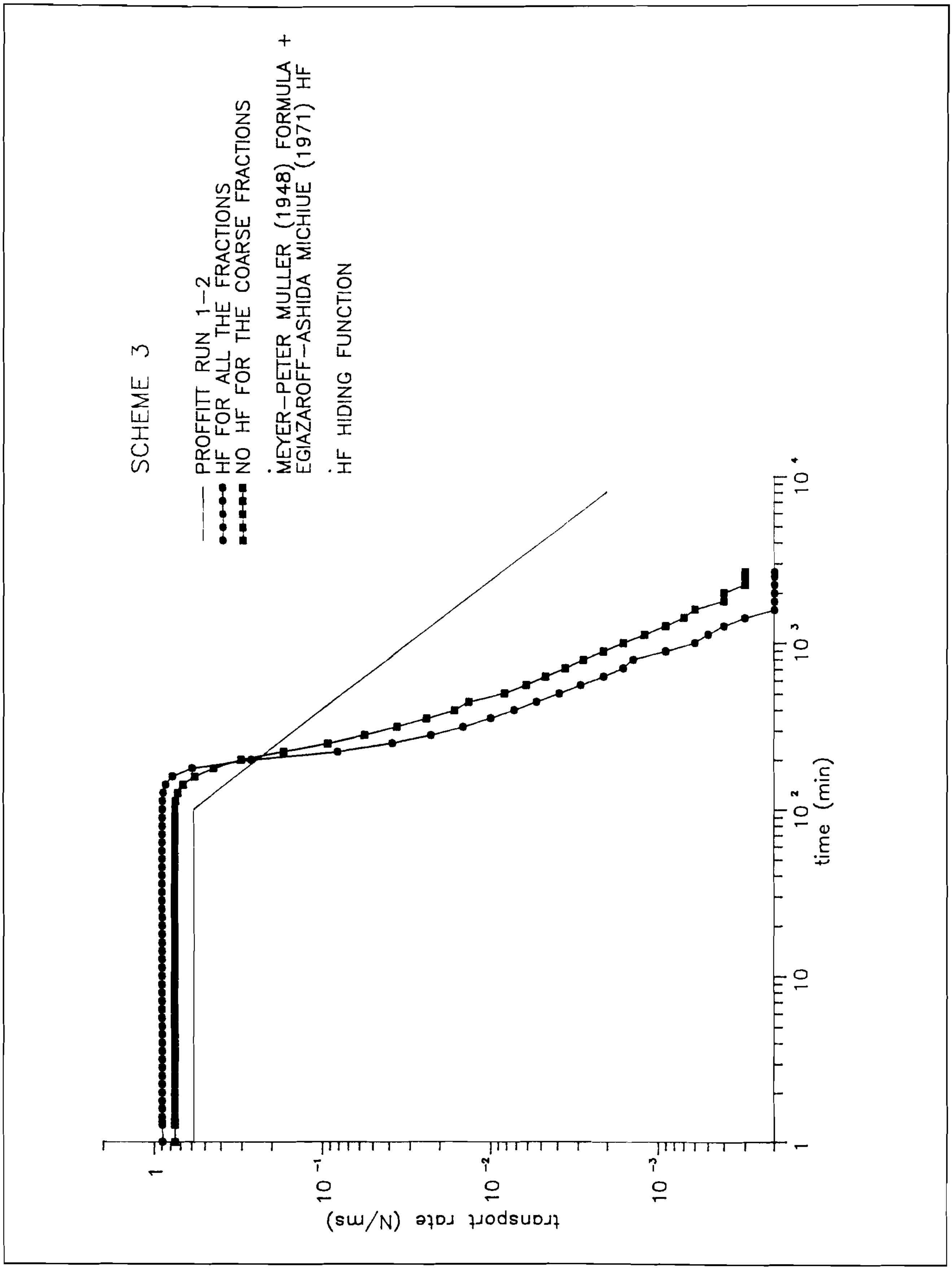

Figure 11 Scheme 3: Numerical solutions with hiding function applied to all sediment sizes

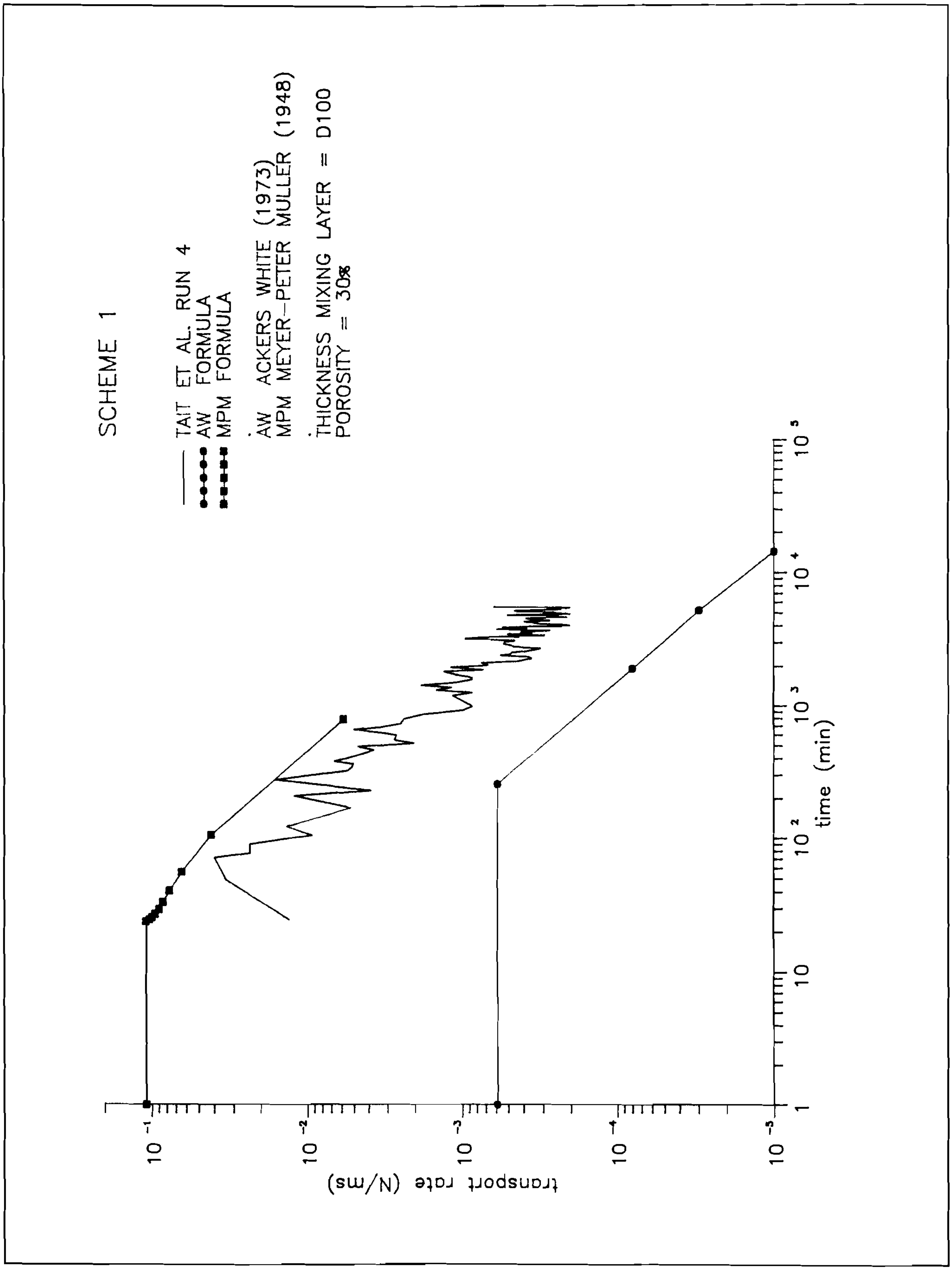

Figure 12 Scheme 1: Total sediment transport rate against time

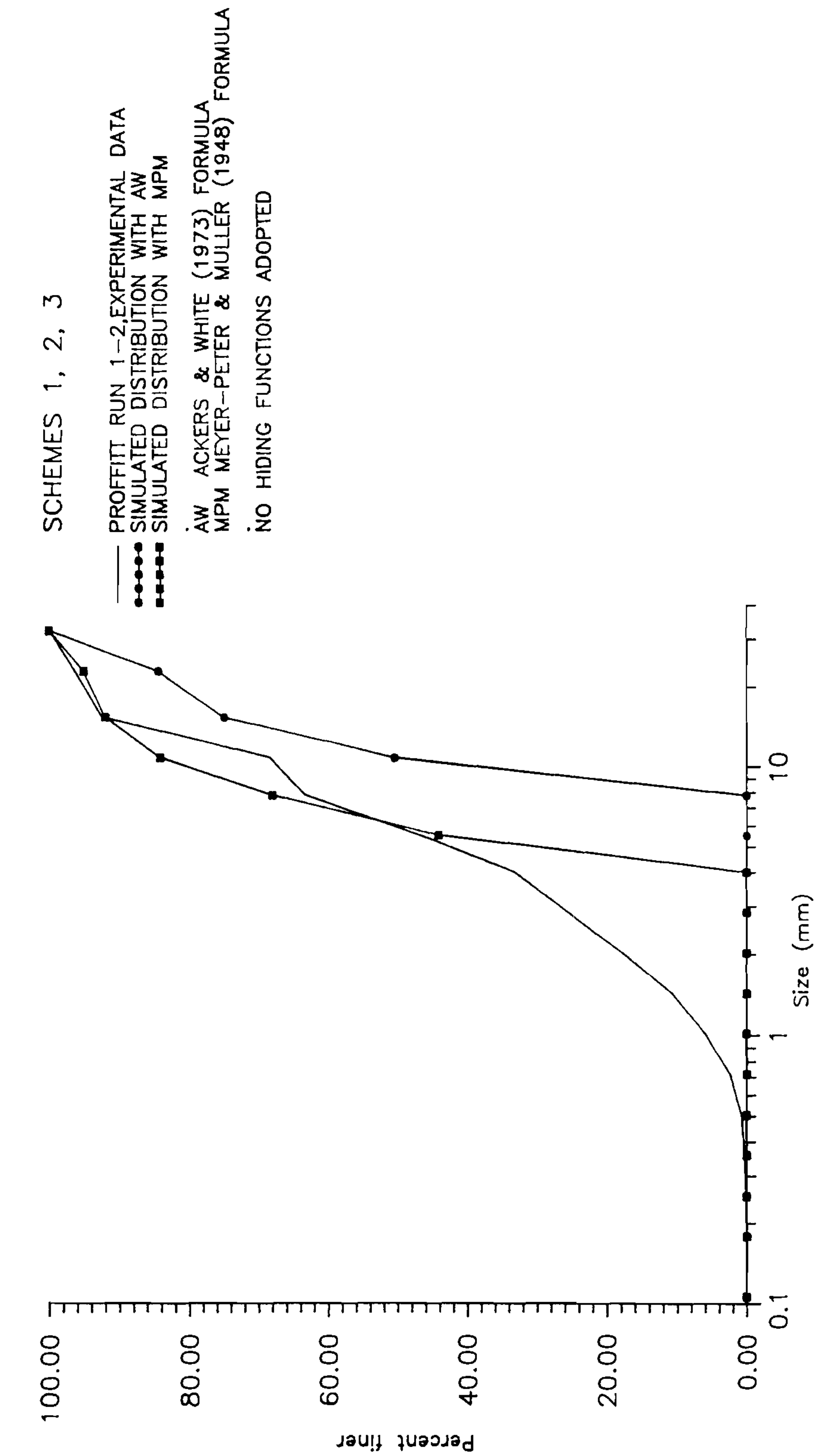

Figure 13 Predicted surface gradings with no hiding

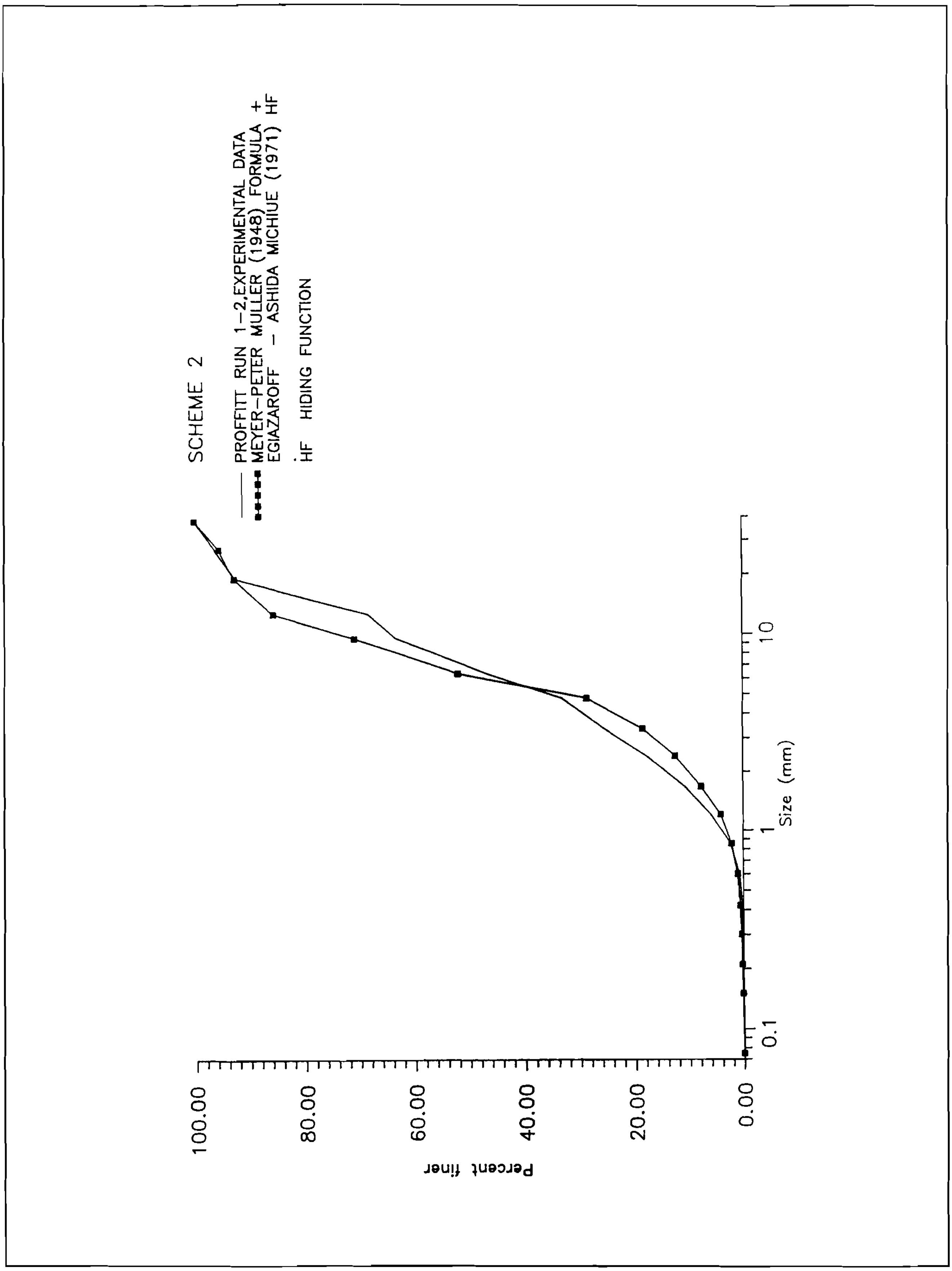

Figure 14 Predicted surface gradings with hiding function

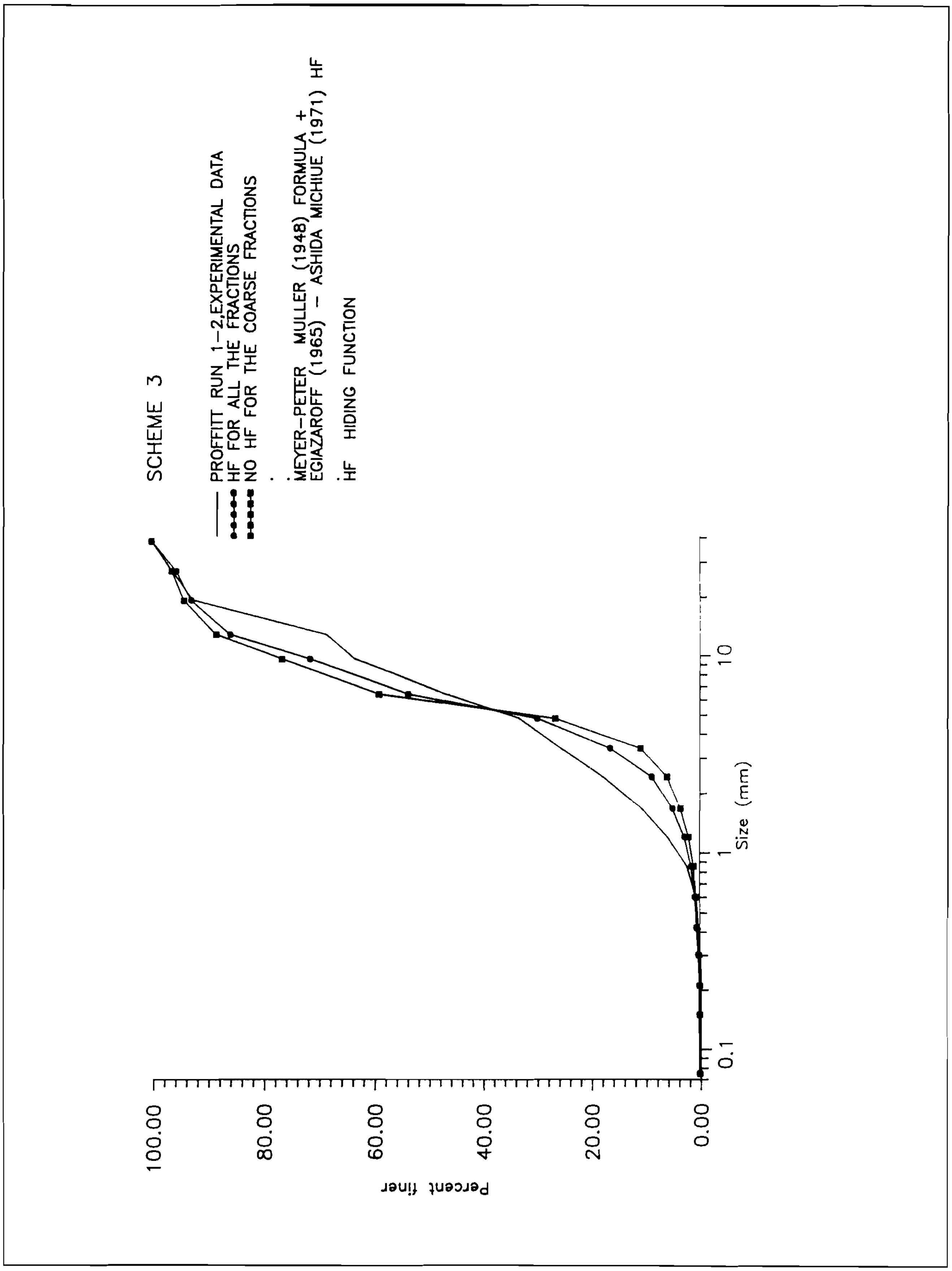

Figure 15 Predicted surface grading with hiding function applied to all sizes

Appendices

Appendix A Meyer-Peter Müller (1948) sediment transport formula

The formula is an empirical relation fitted to experimental data obtained both with uniform and non-uniform sediments.
It gives the value of the bed load for a steady and uniform flow.
A simplified version of the original formula was adopted, referred to the particular case of two-dimensional flow on a plane bed.
The sediment transport rate for unit width by immersed weight q'_s is obtained from the relation:

$$q'_s = 8\rho^{-1/2} (\gamma_s \, . \, D)^{3/2} \left(\frac{\gamma \, . \, J \, .h}{\gamma_s \, . \, D} - 0.047\right)^{3/2} \qquad \text{(A1)}$$

where the density of water is ρ, the specific weights of the water and of the sediment are γ and γ_s, respectively, the size of the sediment is D, the slope is J and the flow depth is h.
The formula was derived from data within the following ranges:

$$\begin{aligned} h &= (1 - 120) \text{ cm} \\ J &= 0.0004 - 0.020 \\ D &= (0.4 - 30) \text{ mm} \\ \gamma_s &= (0.25 - 3.2) \text{ t/m}^3 \end{aligned} \qquad \text{(A2)}$$

When using a non-uniform sediment the characteristic diameter D_{50} was used to describe the mixture.

Appendix B

Ackers & White (1973) sediment transport formula and Day (1980) hiding function

Appendix B. Ackers & White (1973) sediment transport formula and Day (1980) hiding function

This total sediment transport formula is an empirical relation, obtained using experimental data for uniform sediments.
It is based on three dimensionless parameters:

dimensionless grain size:

$$D_{gr} = D \left(\frac{g(-1+\gamma_s/\gamma)}{v^2} \right)^{1/3} , \qquad (B1)$$

particle mobility:

$$F_{gr} = \frac{u_*^n}{(g \cdot D(-1+\gamma_s/\gamma)^{1/2}} \left(\frac{U}{32^{1/2}\mathrm{Log}_{10}(10h/D)} \right)^{1-n} , \qquad (B2)$$

sediment transport:

$$G_{gr} = \frac{X \cdot h}{D \cdot \gamma_s/\gamma} \left(\frac{u_*}{U} \right)^n , \qquad (B3)$$

where D is the dimension of the sediment, g the acceleration due to gravity, γ_s and γ the specific weight of the water and of the sediment,respectively, v the kinematic viscosity of the water, u_* and U the shear and mean velocity of the flow, h the depth of the flow, X the sediment concentration of the sediment by dry weight and n a coefficient defined by (B5), depending on the size of the sediment.

The general sediment transport function is given by the equation:

$$G_{gr} = C \left(\frac{F_{gr}}{A} - 1 \right)^m \qquad (B4)$$

The coefficients A, C and m depend, like n, on the size of the sediment through the parameter D_{gr}.

If $D_{gr} > 60$:

$n = 0$
$A = 0.17$
$m = 1.5$
$C = 0.025,$

(B5)

if $1 < D_{gr} < 60$:

$$n = 1 - 0.56 \, Log_{10} D_{gr}$$

$$A = \frac{0.23}{D_{gr}^{1/2}} + 0.14$$

$$m = \frac{9.66}{D_{gr}} + 1.34$$

$$Log_{10}C = 2.86 Log_{10}D_{gr} - (Log_{10}D_{gr})^2 - 3.53.$$

Using experimental data obtained with non uniform sediments, Day proposed a modification to the equations in which the equation (B4) is replaced by the following equation, applied to each sediment size class of the mixture:

$$G_{gr}' = C \left(\frac{F_{gr}}{A'} - 1 \right)^m \qquad \text{(B6)}$$

G_{gr}' expresses the transport rate of the sediment with characteristic diameter D.
For that size class the actual transported material depends on its availability in the flow and is, theoretically, obtained by multiplying G_{gr}' time the ratio between its weight and that of all the sediment in the flow.
As this ratio is unknown, in practice, the actual transported material is given by the product of G_{gr}' time the value of the fraction of this size class in the bed.

The function A′, appearing in equation (B6), is a function of the grading of the mixture and of the size considered:

$$\frac{A}{A'} = 0.4 \left(\frac{D_u}{D} \right)^{-0.5} + 0.6 \,, \qquad \text{(B7)}$$

being D_u a parameter depending on the mixture through its characteristics diameters D_{16}, D_{50}, D_{84}:

$$\frac{D_u}{D_{50}} = 1.62 \left(\frac{D_{84}}{D_{16}} \right)^{-0.28} \qquad \text{(B8)}$$

The formula proposed was obtained for $D/D_u < 4$ and, therefore, its extension to greater values contains some uncertainty.

Appendix C

Linear waves

Appendix C Linear waves

An equation of the form:

$$\frac{\partial m(x,t)}{\partial t} + c\,\frac{\partial m(x,t)}{\partial x} = 0 \qquad \text{(C1)}$$

is a differential equation of the first order and first degree in the unknown variable m(x,t).
It represents a linear wave, moving in the positive direction of the x axis with celerity c.
The state of the variable m(x,t) in t=0 is changed, when reached by the wave, into the state defined by the boundary conditions in x=0.

If the initial conditions are:

$$m(x,0) = f(x) \qquad \text{(C2)}$$

and the boundary conditions are:

$$m(0,t) = g(t) \qquad \text{(C3)}$$

then the solution of the equation is:

$$\begin{aligned} m(x,t) &= f(x-ct) \quad \text{if } t \le x/c \\ m(x,t) &= g(t-x/c) \quad \text{if } t \ge x/c \end{aligned} \qquad \text{(C4)}$$

The relations (43a) were obtained from this general solution.
Further details can be found in Morris & Brown (1964).

Appendix D

Analytical solution of SCHEME2

Appendix F

Data from Tait et al. run 4

Appendix F Data on Tait et al. run 4

Tait et al. run 4

bed sediment grading

grain size	mass collected (g)
6.300	12.96
5.000	124.34
3.350	375.90
2.360	458.85
1.700	487.47
1.180	403.24
0.850	136.28
0.600	54.45
0.425	27.08
0.300	16.47
0.212	15.23
0.150	12.20
0.105	10.57
0.063	13.81

γ_s = 2650 kg/m^3

Value of the hydraulic and geometric quantities

unit discharge	0.030 m^2/s
water depth	0.062 m
bed slope	0.400 %
shear velocity	0.042 m/s
mean velocity	0.480 m/s
flume length	10.00 m