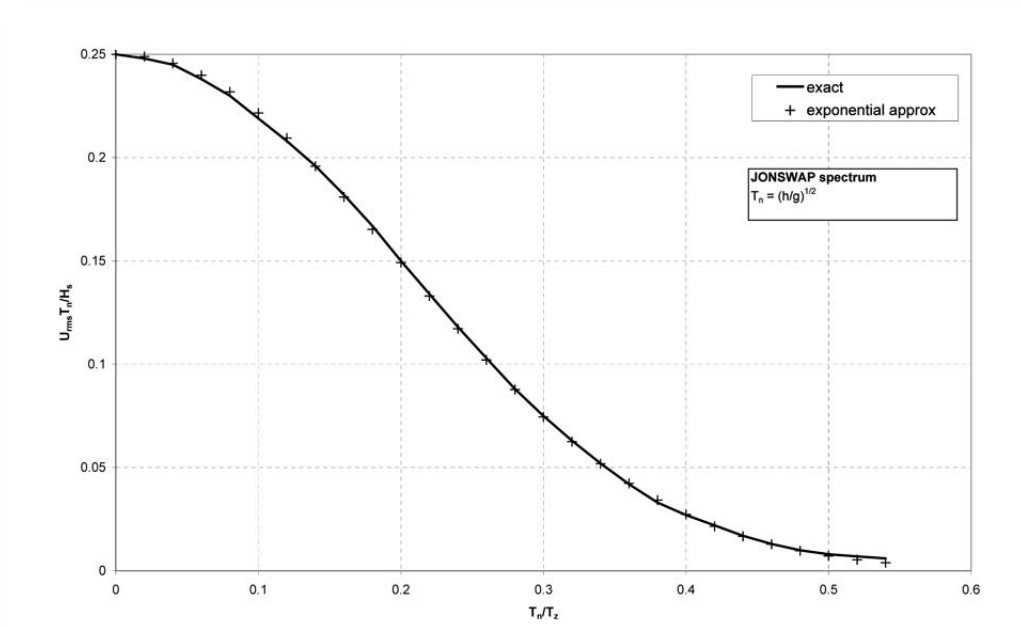


Simplified calculation of wave orbital velocities



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R L Soulsby



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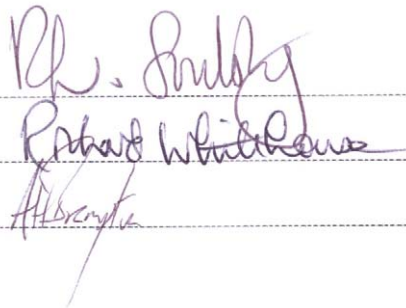
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Summary

Simplified calculation of wave orbital velocities

R L Soulsby

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March 2006

Many coastal problems require the calculation of wave-induced orbital velocities at the sea-bed. Various methods for calculating orbital velocities, including some new approximations, are given in this report, providing an optional trade-off between simplicity and accuracy. The simplest methods are well suited to use in exploratory spreadsheets and desk studies, while the more accurate methods are suited to numerical models and detailed design calculations. The cases covered include monochromatic (regular) waves, and irregular (JONSWAP spectrum possibly with directionally spreading) waves without currents, plus monochromatic waves in the presence of a current. The conclusions give a guide to the choice of most appropriate method for particular purposes.

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1. Introduction

Many coastal problems require the calculation of wave-generated oscillatory (orbital) velocities at the sea-bed for applications such as sediment mobility, transport and suspension, bed protection measures, and forces on structures. This is most commonly performed by using linear wave theory to transform the wave height and period to the orbital velocity in a given water depth as described by e.g. Sleath (1984). Only non-breaking waves are considered in this report.

The use of linear wave theory to obtain orbital velocities is justified by the experimental work of Kirkgöz (1986), who found that linear theory gave reasonable agreement with observed orbital velocities under wave crests over his entire range of parameter settings, even at the transformation point of plunging breakers where higher-order theories might be expected to give significantly better results. The velocity under wave troughs was significantly smaller than the predicted linear-theory value.

Depending on the complexity of the problem, either a regular (monochromatic) wave or irregular (spectral) waves may be considered. However, the expression for the orbital velocity amplitude cannot be written explicitly in terms of depth, wave height and period, so indirect methods must be used. This report extends earlier work by Soulsby and Smallman (1986) and Soulsby (1987) to give calculation methods which are simple enough to be written in a single cell of a spreadsheet, for ease of use in practical applications using spreadsheet methods. It is less accurate than some methods, but is adequate to give at least a reasonable estimate for many desk-study applications.

Despite enormous increases in computer speed and power since 1986, it is still often a limitation when using fine grids, long-term simulations, multiple sensitivity tests, or stochastic simulations in present day studies. Efficiency of repeated computations is thus still a desirable goal, especially when wave orbital velocities need to be calculated at every grid point and every time-step of a numerical model. The simple methods may therefore sometimes be preferred to more accurate methods even in numerical models.

2. Regular (monochromatic) waves

Following Soulsby and Smallman (1986), consider a wave of amplitude $a = H/2$ and radian frequency $\omega = 2\pi/T$, where H and T are the wave height and period respectively, which gives rise to a maximum orbital velocity U_w at the sea-bed (or, more correctly, just outside the thin wave boundary layer near the bed). Then U_w is obtained using small-amplitude linear wave theory from

$$\frac{U_w}{a} = \frac{\omega}{\sinh(kh)} \quad (1)$$

In the absence of a steady current, the wavenumber k ($= 2\pi/\text{wavelength}$) is related to the frequency ω by the dispersion relation

$$\omega^2 = gk \tanh(kh) \quad (2)$$

where g is the acceleration due to gravity and h is the water depth. Define dimensionless variables:

$$x = \frac{\omega^2 h}{g} \quad (3)$$

$$y = kh \quad (4)$$

$$F_m = \frac{U_w^2 h}{a^2 g} \quad (5)$$

Then Equation (1) becomes, after use of Equation (2),

$$F_m = \frac{2y}{\sinh(2y)} \quad (6)$$

and the dispersion relation, Equation (2), becomes

$$x = y \tanh y \quad (7)$$

The dimensionless transfer function F_m cannot be written explicitly in terms of x , and hence in terms of H and T , because the dispersion relation, Equation (7), cannot be written explicitly as $y(x)$. However as Equation (7) gives a one-to-one correspondence between x and y , we see from Equation (6) that F_m is a parametric function of x alone. Both F_m and x contain only the known quantities H , T , h and g , and the required quantity U_w . Thus a plot of F_m versus x (obtained by using y as a parameter in Equations (6) and (7)), allows U_w to be obtained directly from the known quantities. For small values of x (shallow-water waves) the value of F_m tends to one, and F_m decreases monotonically with x until it becomes very small for $x > 4$ (deep-water waves).

The quantities F_m and x are unnecessarily complicated for practical calculations, as they contain the squares of the quantities of interest and also contain some unnecessary constants. We therefore define more readily usable quantities by first introducing the natural scaling period T_n defined by

$$T_n = \left(\frac{h}{g} \right)^{\frac{1}{2}} \quad (8)$$

Then the required dimensionless quantities are

$$\frac{U_w T_n}{2H} \equiv \frac{F_m^{\frac{1}{2}}}{4} \quad (9)$$

and

$$\frac{T_n}{T} \equiv \frac{x^{\frac{1}{2}}}{2\pi} \quad (10)$$

A plot of $U_w T_n / 2H$ versus T_n / T (Figure 1) can be used directly for obtaining U_w from H , T , g and h .

The problem of calculating U_w centres around finding a method of solving Equation (7) for y as a function of x . Various methods are available, with different degrees of complexity and different levels of accuracy. Once y has been calculated, the orbital velocity amplitude U_w follows by combining Equations (5) and (6):

$$U_w = \frac{H}{2} \left[\frac{2gy}{h \cdot \sinh(2y)} \right]^{1/2} \quad (11)$$

This approach can be called a two-step method (1. solve Equation (7) for y given x , 2. apply Equation (11)). Three two-step methods are described below. It is also possible to find approximations to U_w directly as a function of the input variables H , T , h and g . Four of these one-step methods are described below.

The relationship of $U_w T_n/2H$ as a function of T_n/T can be derived exactly by taking values of y , deriving x from Equation (7) and hence T_n/T from Equation (10), and deriving F_m from Equation (6) and hence $U_w T_n/2H$ from Equation (9). The exact curve derived this way is shown in Figures 1-3 as a solid line, with approximations to it shown as symbols for the one-step methods given in Sections 2.4 to 2.7. The two-step methods given in Sections 2.1 to 2.3 are sufficiently accurate that they are almost indistinguishable from the exact curve, and hence have not been presented.

2.1 NEWTON-RAPHSON ITERATION

Make successive approximations to the solution of Equation (7) as follows:

$$\text{Set } y_0 = x^{1/2} \quad \text{if } 0 \leq x < 1 \quad (12a)$$

$$\text{or } y_0 = x \quad \text{if } x \geq 1 \quad (12b)$$

Calculate successive values y_1, y_2, y_3, \dots using the iterative Newton-Raphson formula:

$$t_k = \tanh(y_k) \quad (13)$$

$$y_{k+1} = y_k - \frac{y_k t_k - x}{t_k + y_k (1 - t_k^2)}, \quad k = 0, 1, 2, \dots \quad (14)$$

The relative error in y_1 is less than 3×10^{-3} , in y_2 is less than 10^{-6} , and in y_3 is less than 10^{-13} , for all x . The relative error in U_w is: $<0.15\%$ when derived from y_1 , $<3 \times 10^{-5}\%$ when derived from y_2 , and $<3 \times 10^{-12}\%$ when derived from y_3 . The errors are largest for values of x between 0.5 and 2 and tend to zero for very small x and very large x . The number of iterations can be specified for the required accuracy from the above information, making it unnecessary to check for convergence.

2.2 HUNT'S METHOD

Hunt (1979) proposed an approximation formula for $y(x)$:

$$b = 1 / (1 + x(0.66667 + x(0.35550 + x(0.16084 + x(0.06320 + x(0.02174 + x(0.00654 + x(0.00171 + x(0.00039 + x \cdot 0.00011)))))))) \quad (15)$$

$$y = (x^2 + bx)^{1/2} \quad (16)$$

This is computationally fast because it avoids calculations of $\tanh y$. The relative error in y is less than 10^{-5} for all x . The relative error in U_w is $<0.02\%$ for all x . The error is largest for values of x between 1 and 10, and tends to zero for small and large x .

2.3 GILBERT'S METHOD

G. Gilbert (personal communication) derived an analytical approximation:

$$y = x^{1/2} (1 + 0.2x) \quad \text{for } x \leq 1 \quad (17a)$$

$$y = x[1 + 0.2 \exp(2 - 2x)] \quad \text{for } x > 1 \quad (17b)$$

The relative error in y is less than 0.01 for all x . The relative error in U_w is $<1\%$ for all x . The error is largest for values of x between 1 and 5, and tends to zero for small and large x .

2.4 NIELSEN'S METHOD

Nielsen (1985) used a series expansion to obtain the following one-step approximation to the orbital velocity, which, when written out fully, is given by:

$$U_w = 0.5H(g/h)^{0.5} \left[1 - \frac{1}{3} \left(\frac{2\pi T_n}{T} \right)^2 \right] \quad \text{for } \frac{T_n}{T} < 0.20 \quad (18)$$

The comparison of Equation (18) with the exact curve is shown in Figure 1. Equation (18) is accurate to within 1% for $T_n/T < 0.20$ (the limit of validity stated by Nielsen). If it is extrapolated beyond that limit of T_n/T (deeper water or higher frequencies) it rapidly becomes a serious under-estimate (even negative), so the limit should be strictly observed. Although this method is very accurate for small T_n/T , it is desirable to have methods that can be used over the entire practical range of T_n/T .

2.5 SOULSBY AND SMALLMAN METHOD

Soulsby and Smallman (1986) gave the following one-step approximation to the transfer function $F_m^{1/2}$, without having to go via the dispersion relation:

$$F_m^{1/2} = (1 - 0.670x + 0.110x^2)^{1/2} \quad \text{for } 0 \leq x < 1 \quad (19a)$$

$$= 1.72 x^{1/2} \exp(-0.9529x) \quad \text{for } 1 \leq x < 3.2 \quad (19b)$$

$$= 2 x^{1/2} e^{-x} \quad \text{for } x \geq 3.2 \quad (19c)$$

U_w can then be obtained from Equation (9). The relative error is less than 1% for all x .

2.6 SOULSBY COSINE APPROXIMATION

A one-step method was devised recently to allow the orbital velocity to be calculated to a reasonable approximation within a single cell of a spreadsheet. This is useful when calculating tables of results for wave-induced forces, sediment transport, etc. in spreadsheet form. Figure 2 shows a plot of $U_w T_n/2H$ versus T_n/T , with the "exact" curve approximated by a cosine in the range $0 \leq T_n/T \leq 0.4$. This leads to the approximation formula

$$U_w = 0.25 H (g/h)^{1/2} [1 + \cos(8.055\xi)] \quad \text{for } \xi < 0.4 \quad (20a)$$

$$= 0 \quad \text{for } \xi \geq 0.4 \quad (20b)$$

where

$$\xi = \left(\frac{h}{gT^2} \right)^{\frac{1}{2}} = \frac{T_n}{T} \quad (21)$$

The approximation curve gives relative errors in the orbital velocity with a maximum *under*-estimate of 3.3% at $\xi = 0.14$ (longish period waves) and a maximum *over*-estimate of 18% at $\xi = 0.29$ (shortish period waves). At values of $\xi > 0.34$ the relative under-estimate is large, but since the velocities generated by these short-period, deep-water waves will be very small, the error is less important.

2.7 SOULSBY EXPONENTIAL APPROXIMATION

Subsequent to obtaining the cosine approximation, it has been found that a more accurate fit can be obtained by using an exponential approximation (Figure 3, crosses):

$$U_w = \left(\frac{H}{2} \right) \left(\frac{g}{h} \right)^{\frac{1}{2}} \exp \left\{ - \left[\frac{4.41}{T} \left(\frac{h}{g} \right)^{\frac{1}{2}} \right]^{2.45} \right\} \quad (22)$$

This method is accurate to better than $\pm 1.1\%$ for $T_n/T < 0.21$, and hence gives a similar accuracy to the Nielsen method ($\pm 1\%$ for $T_n/T < 0.20$) in this range. For $T_n/T > 0.21$, Equation (22) gives an increasingly large relative over-estimate, but it can be used reasonably satisfactorily for all depths and periods because it tends asymptotically to zero. Equation (22) has a simplifying advantage over Equation (20) for spreadsheet use, in that a single equation covers all inputs. It also gives non-zero velocities for $T_n/T > 0.4$.

The coefficients 4.41 and 2.45 were chosen as a compromise giving greater accuracy for small ξ , which usually correspond to the largest velocities. Equation (22) over-estimates by $>10\%$ for $\xi > 0.26$. However, the velocities for $\xi > 0.26$ will usually be very small so that errors are less important. For cases of deep water or short-period waves a better fit is obtained by using values of the coefficients of 4.45 and 2.75 instead of 4.41 and 2.45 in Eq (22). This gives a better accuracy for $T_n/T > 0.2$ (Figure 3, circles).

For some applications, particularly using spreadsheets, methods 2.6 or 2.7 will be sufficiently accurate. For applications requiring greater accuracy the methods described in Sections 2.1 to 2.3 can be used.

3. Irregular (spectral) waves

Under natural conditions the wave climate is represented by a spectrum of waves of different frequencies, amplitudes and directions. In many cases the only parameters which are known about the sea-conditions are the significant wave height H_s and the zero-crossing period T_z . The best that can then be done is to fit a realistic surface elevation spectrum $S_n(\omega)$ to these two parameters. One of the most widely accepted two-parameter spectra is the JONSWAP spectrum (Hasselmann et al, 1973), given by

$$S_{\eta}(\omega) = 2\pi\alpha g^2 \omega^{-5} \exp \left\{ -\frac{5}{4} \left(\frac{\omega}{\omega_p} \right)^{-4} \right\} \gamma^{\psi(\omega)} \quad (23)$$

where

$$\psi(\omega) = \exp \left\{ -\frac{(\omega - \omega_p)^2}{2\beta^2 \omega_p^2} \right\} \quad (24)$$

Here ω_p is the radian frequency at the peak of the spectrum, γ and β are constants, and α is a variable which depends on the wind-speed and duration. We use the standard values of the constants, $\gamma = 3.3$ and $\beta = 0.07$ for $\omega < \omega_p$, $\beta = 0.09$ for $\omega > \omega_p$. The variables α and ω_p can be related to H_s and T_z respectively, so that a particular sea-state described only by H_s and T_z corresponds to a particular JONSWAP spectrum.

Following the method described by Soulsby and Smallman (1986) and Soulsby (1987), the root-mean-square orbital velocity U_{rms} at the sea-bed can be computed by summing the velocity contributions from each frequency (derived by linear wave theory) over the whole range of frequencies. This can be presented in a similar way to that given for monochromatic waves in Section 2, but now plotting $(U_{rms} T_n)/H_s$ as a function of (T_n/T_z) in terms of the spectral parameters H_s and T_z . As before, $T_n = (h/g)^{1/2}$. The computed values were tabulated by Soulsby and Smallman (1986). An approximation to this curve presented by Soulsby and Smallman (1986), and a new, simpler, approximation devised for use in spreadsheet applications, are presented below. Soulsby (1987) also treated the alternative spectral forms of Pierson-Moskowitz, Bretschneider, ISSC and ITTC, but these are not dealt with here because the JONSWAP form is more appropriate for the limited-depth applications that usually apply for sediment transport and bed protection studies.

3.1 SOULSBY AND SMALLMAN METHOD

Soulsby and Smallman (1986) presented the approximation formula

$$\frac{U_{rms} T_n}{H_s} = \frac{0.25}{(1 + At^2)^3} \quad (25)$$

where

$$A = \left[6500 + (0.56 + 15.54t)^6 \right]^{1/6} \quad (26)$$

and

$$t = \frac{T_n}{T_z} = \frac{1}{T_z} \left(\frac{h}{g} \right)^{1/2} \quad (27)$$

This approximation fits the exact computed values to an accuracy of better than 1% in the range $0 \leq T_n/T_z \leq 0.54$. Velocities are very small for $T_n/T_z > 0.54$.

3.2 SOULSBY EXPONENTIAL APPROXIMATION

A slightly less accurate, but rather simpler, approximation has been devised to allow the r.m.s. orbital velocity from a JONSWAP spectrum to be calculated with a single cell of a spreadsheet. This uses the following exponential approximation (Figure 4):

$$U_{\text{rms}} = \left(\frac{H_s}{4} \right) \left(\frac{g}{h} \right)^{1/2} \exp \left\{ - \left[\frac{3.65}{T_z} \left(\frac{h}{g} \right)^{1/2} \right]^{2.1} \right\} \quad (28)$$

The accuracy of this fit is rather better than the fit presented for monochromatic waves in Section 2.7. The accuracy of Equation (28) is as follows:

relative error	< 1.2% <i>over-estimate</i> for $0 \leq t < 0.14$
	< 1.0% <i>under-estimate</i> for $0.14 \leq t < 0.34$
	< 4% <i>over-estimate</i> for $0.34 < t \leq 0.40$
	< 35% <i>under-estimate</i> for $0.40 < t \leq 0.54$
	> 35% <i>under-estimate</i> for $t > 0.54$,

where t is given by Equation (27). Again, velocities are very small for $T_n/T_z > 0.54$.

For many applications this method will be sufficiently accurate, since the JONSWAP spectrum is itself only an approximation to actual measured spectra. However, applications in deep water, or for short-period waves (large t) might benefit from the more accurate method described in Section 3.1.

4. *Other factors*

4.1 PRESENCE OF A CURRENT

When monochromatic waves propagate in the presence of a depth-uniform current of speed U , the wave dispersion relation (Equation 2) must be modified to:

$$(\omega - Uk \cos \phi)^2 = gk \tanh(kh) \quad (29)$$

The direction ϕ of the current is defined such that $\phi = 0^\circ$ corresponds to a current travelling in the same direction as the wave, and $\phi = 180^\circ$ corresponds to a current travelling in the opposite direction to the wave. Equation (29) applies to any direction ϕ , but it is seen that only the component of current parallel to the direction of wave propagation has an effect on the wave dispersion. Defining this component as

$$U_p = U \cos \phi \quad (30)$$

the non-dimensional form of Equation (29) can be written, by analogy with Equation (7), as

$$\left(x^{1/2} - Fr \cdot y \right)^2 = y \tanh y \quad (31)$$

where x and y are defined by Equations (3) and (4), and

$$Fr = U_p / (gh)^{1/2} \quad (32)$$

is the Froude number for the component of current parallel to the wave direction (and can be negative).

When the current is in the same direction as the waves, the wavelength becomes longer than the no-current case (and the wave height becomes smaller to conserve energy transmission). When the current is in the opposite direction to the waves, the wavelength becomes shorter than the no-current case (and the wave height becomes larger). Opposing currents of speed greater than the phase-speed of the waves prevent the wave from propagating. This applies to the current component parallel to the waves from currents running in any direction in the opposing sector.

The solution of Equation (31) for y (given x and Fr) is much less straightforward than for the no-current case, especially for opposing currents. Even deciding whether an opposing current exceeds the limiting value is not straightforward. Southgate and Oliver (1989) gave details of two methods: a Newton-Raphson iteration method and a look-up table method. They are not repeated here, due to their complexity.

Having solved Equation (31) for y , it is straightforward to obtain the wave orbital velocity by means of Equation (11). The orbital velocity is larger than the no-current case for a following current, and smaller for an opposing current. As an example, consider a wave of height $H = 1\text{m}$ and period $T = 6\text{s}$ in a water depth $h = 5\text{m}$. With no current the orbital velocity $U_w = 0.568\text{ms}^{-1}$. With a following current of 1ms^{-1} this is increased to $U_w = 0.601\text{ms}^{-1}$, whereas with an opposing current of 1ms^{-1} it is decreased to $U_w = 0.511\text{ms}^{-1}$.

The “exact” curves of $U_w T_n / 2H$ versus T_n / T can be derived by starting with y and calculating the corresponding x from Equation (31) and F_m from Equation (6), followed by Equations (9) and (10), in a similar way to the derivation of the “exact” curves in Figures 1 to 3. Figure 5 shows curves derived in this way for Froude numbers of -0.5, -0.2, -0.1, 0, 0.1, 0.2 and 0.5. The relative effect of currents on wave orbital velocities can be gauged from this figure. Froude numbers larger than 0.5 are very rarely encountered in coastal waters. The curves for opposing currents with Froude numbers of -0.2 and -0.5 show that only the longer period waves are able to propagate against the current, and their orbital velocities are strongly reduced from the no-current case (for a given wave height and water depth). The right-most symbol on these curves shows the shortest period wave that can exist with this opposing current. For following currents, it can be seen that a Froude number of only 0.1 is sufficient to more than double the wave orbital velocity for $T_n / T = 0.3$ compared with no current.

Other aspects of wave-current interaction were discussed by Soulsby et al (1993). The extension to an irregular wave spectrum plus a current would involve transforming each wave frequency using the method described here, and combining the velocities. This has not been addressed for this report.

4.2 DIRECTIONAL SPREADING OF A RANDOM SEA

An additional complication of a random sea is that there is an appreciable spread in the wave directions, which is generally expressed by multiplying Equation (23) by a spreading function. For calculations of the wave energy dissipation rate the form of the spreading function can influence the dissipation rate by up to 20% (Brampton et al, 1984). However, because the bottom orbital velocity is related linearly to the surface

elevation, it can be deduced that the relationship between the quantities U_{rms} and H_s is independent of the spreading function.

5. *Conclusions and choice of methods*

When calculating the wave orbital velocity amplitude U_w at the sea bed, a choice must be made between accuracy and simplicity. In many cases, quick and easy methods are required for use in desk studies, or to compare a large number of wave conditions by means of spreadsheets. In these cases, simplified methods are appropriate, provided that the limitations on accuracy are recognised. Simple methods can also be useful for use in gridded computational models, because a reduction in run-time can allow longer-term simulations to be made, or a stochastic approach to uncertainty to be used. In other cases, particularly for final design purposes, as much accuracy as possible needs to be retained. The methods examined in this report allow such choices to be made. They can be summarised as follows:

- Linear wave theory gives a good approximation to the orbital velocity at the sea bed under the wave crest, this being the maximum velocity for force calculations on structures, or for sediment threshold or bed protection stability calculations.
- The reverse orbital velocity beneath the wave trough may be considerably smaller than that under the crest, which can be important when calculating net wave-averaged sediment transport rates.
- For regular (monochromatic) waves, such as are sometimes used in laboratory experiments and which are a good approximation for narrow-banded swell waves, there are several methods available, with a trade-off between accuracy and simplicity.
- For desk studies and spreadsheet use, the Nielsen method (Section 2.4) is accurate to within 1% for waves with period $T > 5(h/g)^{1/2}$, but it should not be used for shorter wave periods.
- The previously developed cosine approximation (Section 2.6) is reasonably accurate for longer period waves, to within 3.3% for waves with $T > 7(h/g)^{1/2}$, but over-predicts at shorter periods. However, the new exponential approximation (below) is generally a better choice, because it is both more accurate and simpler to program into a spreadsheet.
- The new exponential approximation (Section 2.7) gives a similar accuracy ($\pm 1.1\%$) to the Nielsen method for $T > 5(h/g)^{1/2}$ and is better behaved at shorter periods. It over-predicts at these shorter periods, but this is less important because the velocities are usually small. Two sets of coefficients are given, which optimise the fit for either shallow-water or deep-water applications. Either set of coefficients can be used over the whole range of periods and depths without great loss of accuracy.
- For efficient calculations in computational models, Gilbert's method (Section 2.3) and the Soulsby and Smallman method (Section 2.5) are both accurate to within 1% for all input conditions, and are both relatively fast and easy to use.

- For high precision, the Newton-Raphson (Section 2.1) and Hunt (Section 2.2) methods are both very accurate, with a maximum error of 0.02% for the Hunt method, and 0.15% by only one iteration of the Newton-Raphson method (and more accurate for further iterations).
- For most practical applications, waves will generally be specified by spectral parameters such as significant wave height H_s and one (or more) of zero-crossing period T_z , spectral mean period T_m , or peak period T_p . It should be noted that values of wave period derived from the analysis packages of wave measuring devices often yield much less statistically robust values of T_p than of T_m or T_z . This is because they pick the single highest spectral estimate to derive T_p , whereas T_m and T_z are derived from the whole spectrum. It is therefore preferable to use T_m or T_z when calculating the wave orbital velocity. However, the most relevant period for use in force, shear-stress or sediment transport calculations is the peak period, because this is the period at which the energy is centred. For a JONSWAP spectrum, it is a reasonable approximation to take $H_s \approx H_{1/3}$, $T_z \approx T_m$, and $T_p \approx 1.28T_z$.
- The root-mean-square orbital velocity produced by all the waves in a JONSWAP spectrum can be obtained with good accuracy and simplicity from the new exponential approximation for spectral waves given in Section 3.2. This is accurate to $\pm 1.2\%$ for waves with zero-crossing periods $T_z > 2.9 (h/g)^{1/2}$. It is very simple to use in spreadsheets and desk studies.
- For computational models the slightly more complicated, but also more accurate, spectral method of Soulsby and Smallman (Section 3.1) can be used. This is accurate to $\pm 1\%$ for all zero-crossing periods $T_z > 1.8(h/g)^{1/2}$, which covers most cases of practical interest (e.g. $T_z > 2s$ in a depth of 10m, or $T_z > 4s$ in a depth of 40m).
- The effect of a current with a component directed with or against the wave propagation direction can be significant (Figure 5). The size of the effect is determined by the Froude number, $Fr = U_p/(gh)^{1/2}$, where U_p is the current component in the direction of wave propagation. Froude numbers up to 0.1 or 0.2 are not uncommon in the sea, and these can influence the wave orbital velocity by more than $\pm 10\%$ for typical waves and water depths. Hence inclusion of the current can often have a bigger effect on the calculation of orbital velocity than the accuracy of the no-current methods described above. An opposing component of current can completely block wave propagation, particularly for short period waves. The methods described by Southgate and Oliver (1989) should be used for monochromatic waves. Methods of calculating the r.m.s. orbital velocity for a wave spectrum in the presence of a current have not been considered in the present report.
- Directional spreading of a wave spectrum does not affect the r.m.s. orbital velocity.
- Only non-breaking waves are considered in this report, although the methods are commonly (but not fully justifiably) applied to breaking waves as well.
- For sediment transport purposes, many formulae require inputs in terms of monochromatic wave parameters. For such cases, a good approach is to use a spectral method (Section 3.1 or 3.2) to calculate U_{rms} , and then take a monochromatic orbital velocity amplitude of $U_w = \sqrt{2}U_{rms}$. This gives the same

root-mean-square velocity as a sinusoidal wave. The period can be taken as the peak period, but, for statistical stability, preferably calculated as $T_p = 1.28T_z$.

- The relative errors associated with the various methods of calculating wave orbital velocity presented in the report are generally much smaller than the measurement errors of wave heights and periods, and even the value of g may vary with latitude by a few parts per thousand.

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Figures

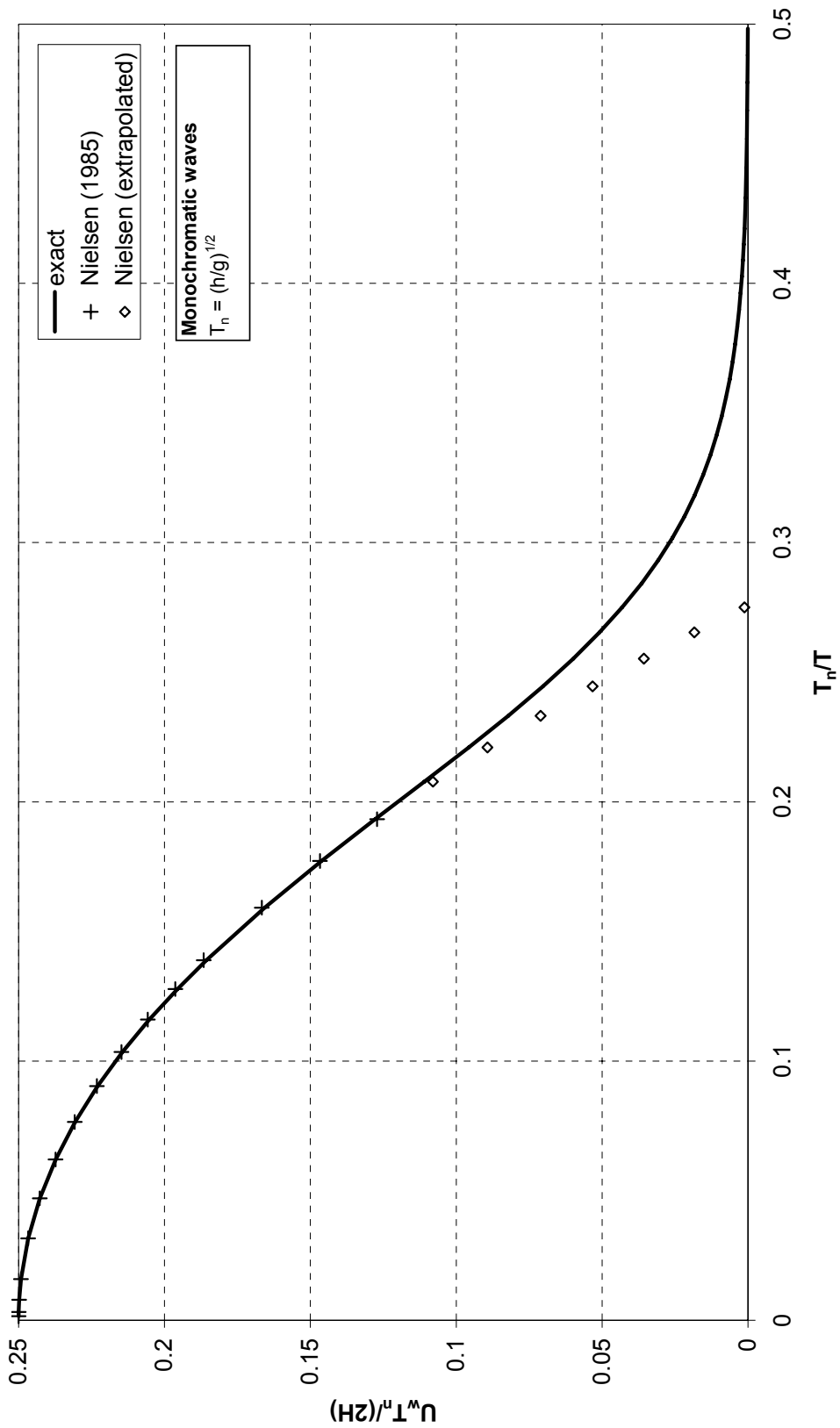


Figure 1 Bottom orbital velocity U_w for monochromatic waves of height H and period T . Exact curve, Nielsen's approximation and extrapolation of Nielsen's approximation (Eq. 18)

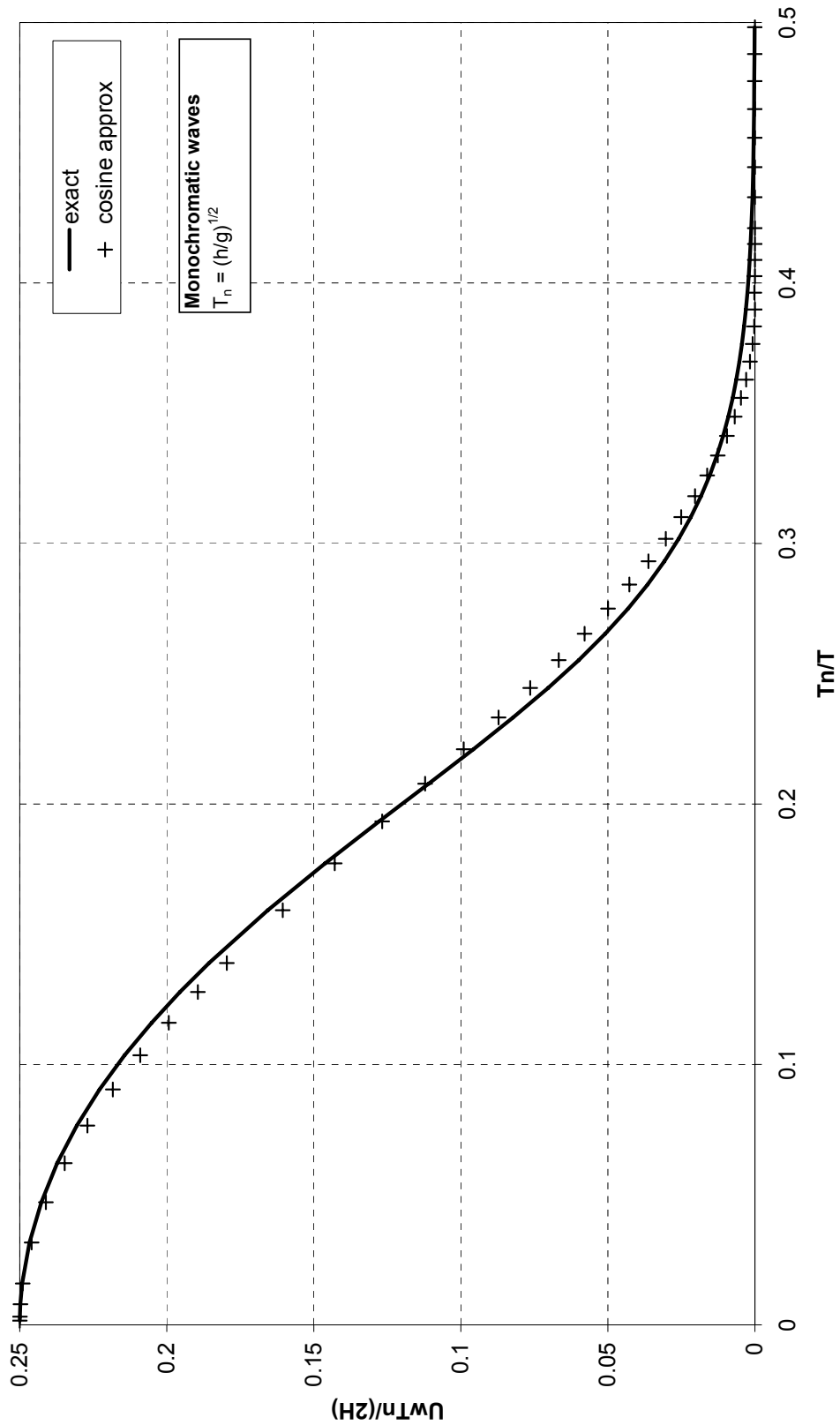


Figure 2 *Bottom orbital velocity U_w for monochromatic waves of height H and period T . Exact curve and cosine approximation (Eq. 20)*

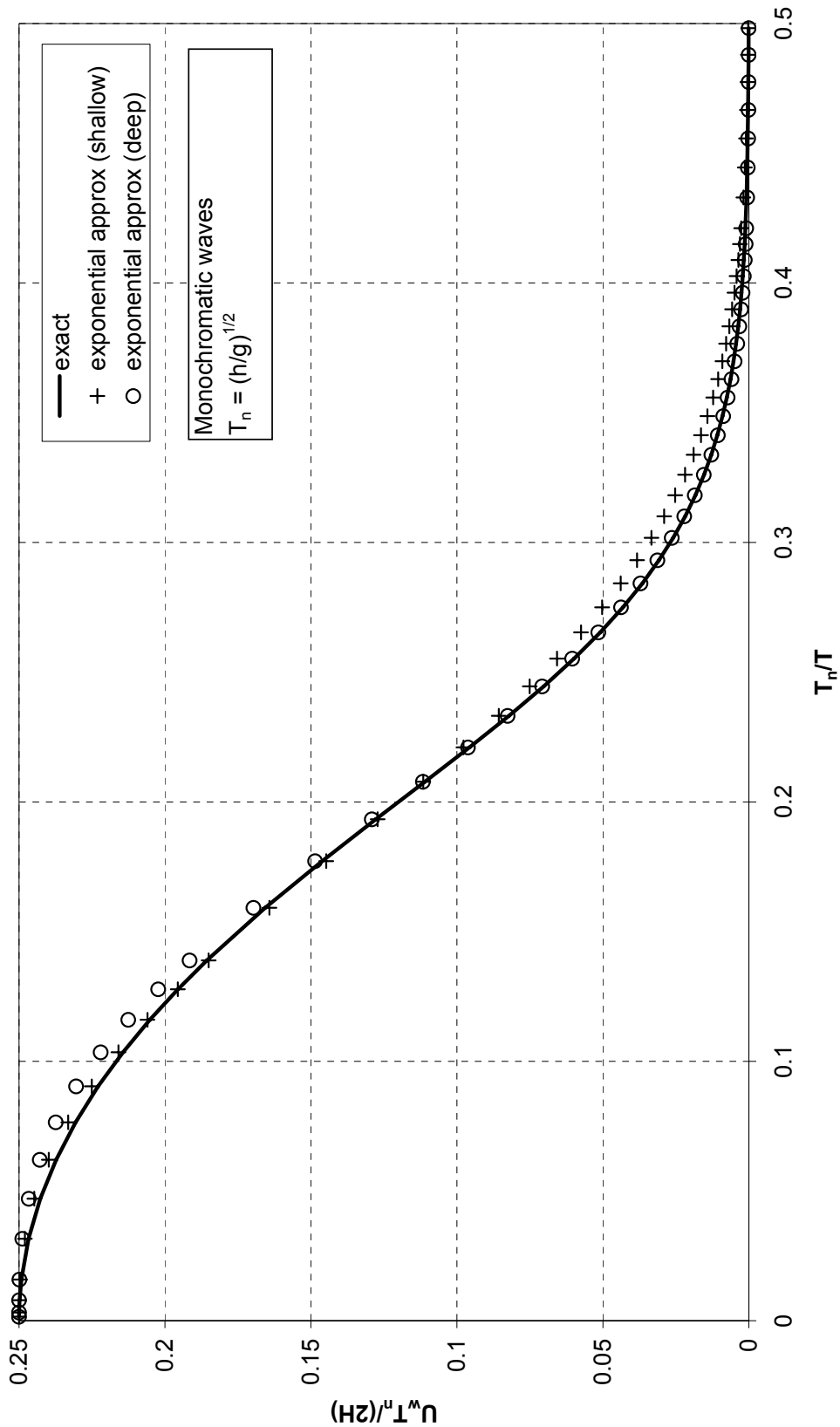


Figure 3 *Bottom orbital velocity U_w for monochromatic waves of height H and period T . Exact curve and exponential approximation (Eq. 22) with coefficients adjusted to fit for (a) shallow water (crosses), (b) deep water (circles)*

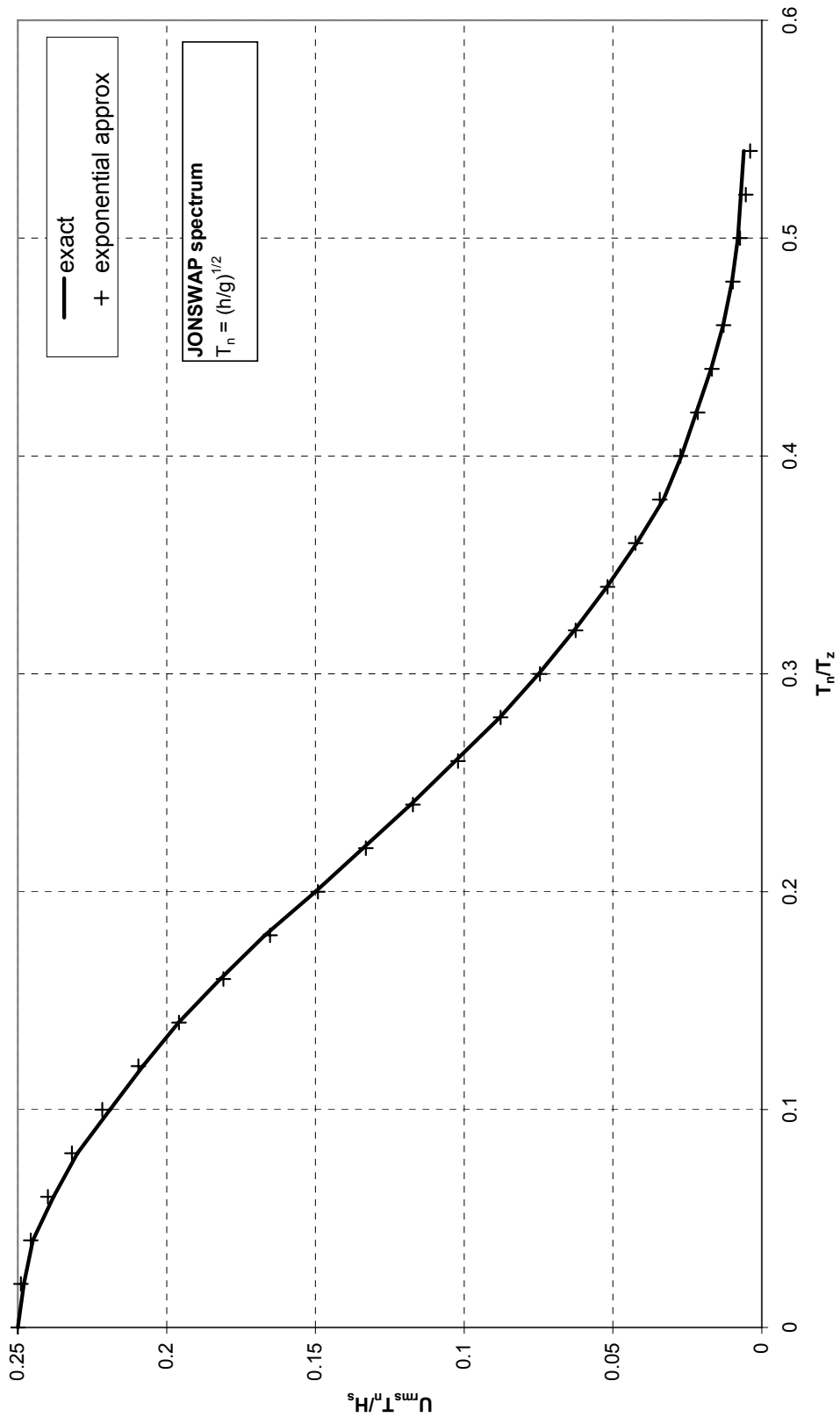


Figure 4 Root-mean-square bottom orbital velocity U_{rms} for a JONSWAP spectrum with significant wave height H_s and zero-crossing period T_z . Exact curve and exponential approximation (Eq. 28)

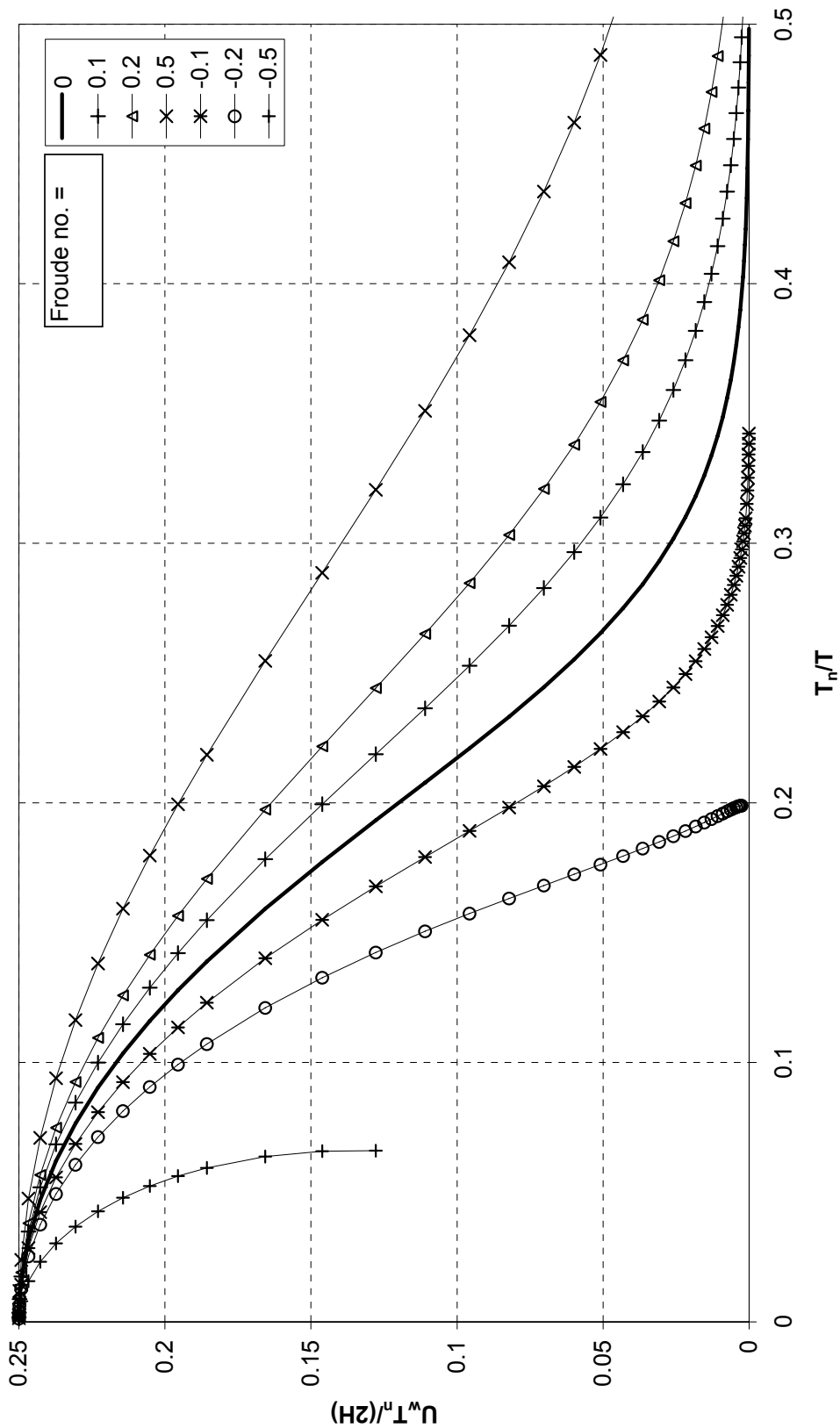
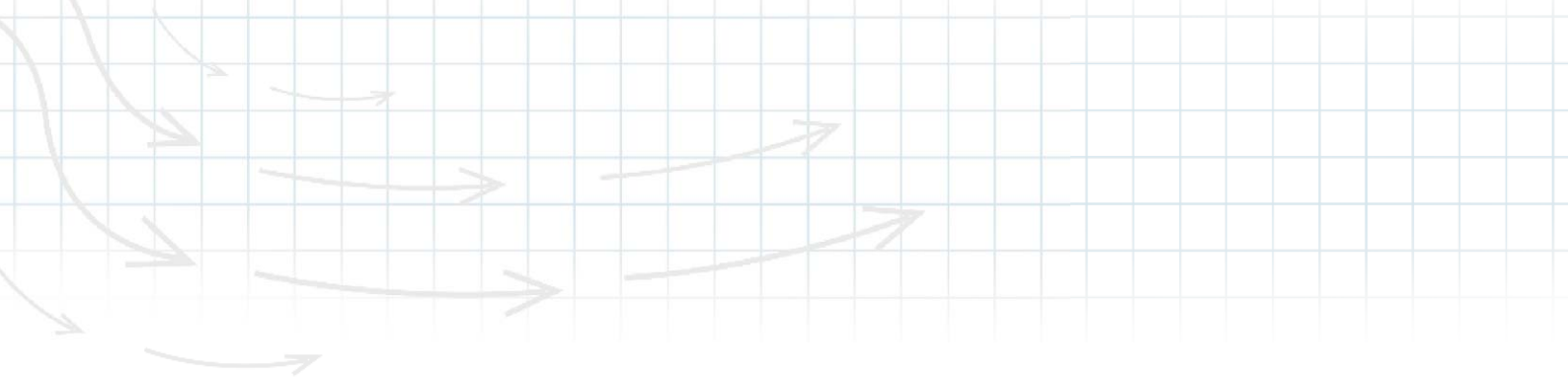


Figure 5 Bottom orbital velocity U_w for monochromatic waves of height H and period T in the presence of a current component with Froude number $Fr = U_p/(gh)^{1/2}$. Exact curves from Eq. 31



HR Wallingford
Working with water

HR Wallingford Ltd
Howbery Park
Wallingford
Oxfordshire OX10 8BA
UK

tel +44 (0)1491 835381
fax +44 (0)1491 832233
email info@hrwallingford.co.uk

www.hrwallingford.co.uk

