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EXTREMES OF A GENERAL GAUSSIAN PROCESS

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ABSTRACT

Assumptions that the sea surface can be represented by a train of regular waves, or can be characterised by a design wave deterministic in form defined by wave height and length, lead directly to definite values of maximum force or deflection for a structure introduced into that sea state.

Recognising that the sea surface is a random process prevents such definite quantities being determined. Rather the designer will want certain extreme values which have only a small chance of being exceeded during the lifetime of the structure.

Methods for describing the frequency of exceedance of extreme design levels are well established for simple one-dimensional quantities which are linear transformations of the random sea state.

This report extends this concept of the estimation of extremes by considering a wide range of general problems where the quantity for which one is seeking the extremes can be defined by an explicit combination of quantities which themselves are derivable by linear transformation from the random sea surface.

The method gives extreme time domain statistics, the inputs to the problem being frequency domain information such as spectral and cross-spectral moments between the various component quantities.

The theoretical development is illustrated by three examples:

(i) estimation of the extremes of the resultant of two orthogonal gaussian components

(ii) a hydraulic actuator which has end stop conditions defined in terms of both displacement and velocity

(iii) estimation of the extreme force on a structure using the Morison method of description in a random sea state.

CONTENTS

INTRODUCTION

Design procedures for structures in the maritime environment have traditionally assumed that the maximum loading or displacement have arisen due to the most severe 'design' wave. This wave has been assumed to be of a deterministic form defined by parameters such as wave height and period, and has lead to the determination of a definite value of the maximum of interest.

However, it is more realistic to consider the sea as a random process and then design can no longer be based on definite values. Instead the designer will expect to be given certain extreme values for quantities like force and displacement which have only a small probability of being exceeded in the life time of the structure.

For simple one-dimensional random processes, which are stationary and have a gaussian distribution, the estimation of extremes which may occur in a given time has been established for many years and is now routine. The most simple example is the estimation of extreme elevations of the wave maxima. With a knowledge of the standard deviation of the surface elevation (indicating total energy) and a frequency parameter, both of which can be derived from the wave spectrum, then an estimate of the extreme elevation within a particular duration can be made. Note that this procedure takes characteristic parameters derived in the frequency domain to give extreme quantities in the time domain.

This process can be transferred simply to other quantities such as force or motion when they are also one-dimensional and linearly related to the waves. For example this procedure is readily applied to the estimate of inertia and diffraction forces on a simple cylinder in a random sea state.

However, more complicated problems arise even with the simple cylinder in random waves. For example, in short crested seas the inertia or diffraction loading will be characterised by linear force responses in two component directions, but the designer will require estimations of the extremes of the resultant.

Returning to unidirectional seas, if the cylinder is small with respect to the wave length then drag forces must be considered together with inertia forces. Morison's equation is commonly used in this situation. This equation estimates the force by a non-linear combination of two components of force, one depending on velocity, one on acceleration, both of which themselves are known by linear transformation of the sea state. Here the designer will need the extremes of the overall maximum taking both components into account.

These problems are both of the type that are considered in this report. However, the method developed is general, not specific to forces (or displacements) in random seas but is applicable to a wide range of problems where the required quantities are known explicitly in terms of gaussian variables derivable by linear transformation of the sea state.

The report addresses the problem of the extremes of 'short' term statistics when in the context of the maritime environment. That is, methods of estimating the expected extreme events during a single storm (i.e. a stationary process) are given, as opposed to 'long' term statistical methods which take into account the variation of sea state also over the life of the structure.

This type of problem can be cast into the general form of finding the probability that within a given time a point, whose wandering represents an n-dimensional gaussian process, will escape from a certain 'safe' region in its space. The co-ordinate components of the space represent the variables entering the problem which are derivable by linear transformation from the sea-state and the safe region is defined by the explicit expression for the required quantity. With an assumption of independence of escapes the problem reduces to finding the mean frequency of escapes. The method of tackling the basic one-dimensional problem of this type was provided by the classic paper of Rice(1). A two-dimensional extension giving the extremes of the magnitude of a vector was given by Huntington and Gilbert(2). The n-dimensional problem was treated by Veneziano et al(3) but only for uncorrelated components and regions of simple shape.

Here the components are allowed to be intrinsically correlated so that they cannot be uncorrelated simply by transformation. This is a point of some importance since, as will be shown in the examples, it is often necessary in reducing problems to the required form to define the components to be derivatives of the same variable. It will be shown how the required result can be reduced to an integral over the boundary of the safe region. In the general case this cannot be integrated analytically, so it will be further shown how a simple formula can be derived as an asymptotic approximation.

For most purposes, since it is large extreme values which are of interest, this approximation is all that is required.

MEAN FREQUENCY AND PROBABILITY

The quantity to be derived for the general problem will be the mean frequency f_s of outcrossings from the safe region. For example, in the one dimensional case, treated by Rice the mean frequency of upcrossings of a value x is given by:

$$f_s = \frac{1}{2\pi}\left(\frac{m_2}{m_0}\right)^{\frac{1}{2}} e^{-\frac{1}{2}\frac{x^2}{m_0}} \tag{1}$$

where m_n is the nth moment of the spectrum of x. This mean frequency can either be taken itself as an indication of reliability or if it is necessary to deduce the probability of no outcrossing within a given time span T, then the crossings will be assumed, being rare events, to be uncorrelated. They then have a Poisson distribution and the required probability is given by:

$$p = e^{-f_s T}$$

This formula together with (1) gives the usual expression for the calculation of extreme waves. The assumption of independent crossings will err on the safe side, if at all, since for a fixed mean frequency, any tendency for crossings to bunch increases the probability of no event over a given time. Such bunching could occur, for instance, with a small high frequency signal superimposed on a low one or with a very narrow spectrum process giving slow modulations of the primary frequency.

MEAN OUT-CROSSING FREQUENCY

An n-dimensional gaussian process $\underline{x}(t)$ is considered to be safe within a region R of n-space bounded by a surface s. What is the mean frequency of f_s with which outcrossings of s occur?

Let $p(\underline{x},\underline{\dot{x}})$ be the joint probability distribution of displacement and velocity of the process and let $\underline{n}$ be the outward normal at a general point of s. Then within a small time interval δt taken at random the range of normal distance covered will be $(\underline{\dot{x}}.\underline{n})\delta t$. Hence the argument of Rice[1] may be generalized to give the mean outcrossing frequency as:

$$f_s = \int_s \int_\Gamma p(\underline{x},\underline{\dot{x}})\ (\underline{\dot{x}}\cdot\underline{n})\ d\underline{\dot{x}}\ ds \tag{2}$$

where Γ is the region of $\underline{\dot{x}}$ space where $\underline{\dot{x}}\cdot\underline{n} > 0$.

The joint distribution $p(\underline{x},\underline{\dot{x}})$ will be a 2n dimensional gaussian distribution determined by a matrix of correlation coefficients. By rotating and scaling the axes it can always be ensured that the displacements alone have a unit isotropic distribution so that the correlation matrix has the form:

$$\Sigma = \left(\begin{array}{c|c} I & \tau \\ \hline -\tau & \lambda \end{array}\right)$$

where $\lambda_{ij} = \{\dot{x}_i \dot{x}_j\}$

and $\tau_{ij} = \{x_i \dot{x}_j\} = -\tau_{ji}$

In order to eliminate the cross correlations introduce

$$u_i = \dot{x}_i + \tau_{ik} x_k$$

Then

$$\{x_i u_j\} = \{x_i \dot{x}_j\} + \tau_{jk} \{x_i x_k\} = \tau_{ij} + \tau_{ji} = 0$$

and

$$\{u_i u_j\} = \{\dot{x}_i \dot{x}_j\} + \tau_{ik} \{x_k \dot{x}_j\} + \tau_{jk} \{x_k \dot{x}_i\} + \tau_{ik} \tau_{jl} \{x_k x_l\}$$

$$= \lambda_{ij} + \tau_{ik}\ \tau_{kj} + \tau_{jk}\ \tau_{ki} + \tau_{ik}\ \tau_{jk}$$

$$= \lambda_{ij} - \tau_{ik}\ \tau_{jk} = \rho_{ij} \text{ say} \tag{3}$$

Also $\underline{\dot{x}}\cdot\underline{n} = \underline{u}\cdot\underline{n} - \gamma$

$$\text{where } \gamma = \tau_{ik}\ n_i\ x_k \tag{4}$$

Hence, since $\underline{x}$ and $\underline{u}$ are uncorrelated (2) becomes

$$f_s = \left(\frac{1}{2\pi}\right)^{n/2} \int_s e^{-\frac{1}{2}x_i x_i} F(\underline{x})\, ds$$

where $F(\underline{x}) = \int_\Gamma p(\underline{u})\ (\underline{u}.\underline{n} - \gamma)\ d\underline{u}$

But since Γ is the region $\underline{u}.\underline{n} > \gamma$ this is an integral over the whole space except in the direction of $\underline{n}$. Hence it reduces to a single integral which depends only on the variance of $\underline{u}.\underline{n}$. Since this variance is from (3)

$$\sigma^2(\underline{n}) = \rho_{ij} n_i n_j \tag{5}$$

$$F(\underline{x}) = \frac{1}{(2\pi)^{\frac{1}{2}}\sigma} \int_\gamma^\infty e^{-\frac{1}{2} v^2/\sigma^2} (v - \gamma)\, dv$$

$$= \frac{1}{(2\pi)^{\frac{1}{2}}} \left[\sigma\, e^{-\frac{1}{2} \gamma^2/\sigma^2} - \gamma \int_{\gamma/\sigma}^\infty e^{-\frac{1}{2}v^2} dv\right]$$

Hence

$$f_s = \left(\frac{1}{2\pi}\right)^{(n+1)/2} \int_s \left[e^{-\frac{1}{2}(x_i x_i + \frac{\gamma^2}{\sigma^2})} \sigma - \tau_{ij} n_i x_j e^{-\frac{1}{2}x_k x_k} \int_{\gamma/\sigma}^\infty e^{-\frac{1}{2}v^2}\right] ds \tag{6}$$

The integral over v in the second half of (6) can be eliminated by an integration by parts

$$\text{Since}\quad -x_j e^{-\frac{1}{2} x_k x_k} = \frac{\partial}{\partial x_j} e^{-\frac{1}{2} x_k x_k}$$

and τ is antisymetric, any particular pair τ_{ij}, τ_{ji} gives for the second half of (6) an integral of the form

$$\int_s f(x_k)\ \left(n_i \frac{\partial}{\partial x_j} - n_j \frac{\partial}{\partial x_i}\right) e^{-\frac{1}{2} x_k x_k}\ ds$$

$$= \int_s \left(n_i \frac{\partial}{\partial x_j} - n_j \frac{\partial}{\partial x_i}\right)\left[f(x_k) e^{-\frac{1}{2}x_k x_k}\right] ds - \int_s e^{-\frac{1}{2}x_k x_k} \left(n_i \frac{\partial}{\partial x_j} - n_j \frac{\partial}{\partial x_i}\right) f(x_k)\, ds$$

But $\int_s (n_i \frac{\partial}{\partial x_j} - n_j \frac{\partial}{\partial x_i})\, h\,(x_k)\, ds = 0$

for a closed surface S or one extending to infinity since in two-dimensions it becomes $\int_s \frac{dh}{ds}\, ds$ and in higher dimensions by first holding the other variables constant and then integrating over them it becomes an integral of such two-dimensional cases.

Also $\frac{\partial}{\partial x_j} \int_{\gamma/\sigma} e^{-\frac{1}{2}v^2} = - e^{-\frac{1}{2}\gamma^2/\sigma^2} \frac{\partial}{\partial x_j} (\frac{\gamma}{\sigma})$

Hence putting these results together (6) simplifies to

$$f_s = (\frac{1}{2\pi})^{(n+1)/2} \int_s e^{-\frac{1}{2}[x_i x_i + \frac{(\tau_{ij} n_i x_j)^2}{\sigma^2(\underline{n})}]} \{\sigma(\underline{n}) + \tau_{ij} n_i \frac{\partial}{\partial x_j} [\frac{\tau_{kl} n_k x_l}{\sigma(\underline{n})}]\}\, ds \tag{7}$$

where $\sigma^2(\underline{n}) = \rho_{ij} n_i n_j$

The required mean frequency is thus reduced to evaluating an integral over the boundary of the safe region.

LINEAR AND SPHERICAL BOUNDARIES

For a linear boundary s the integral (7) can be evaluated. Suppose first of all that the boundary is $x_1 = r$ then since $\underline{n}$ is constant over s, (7) becomes:

$$f_s = (\frac{1}{2\pi})^{(n+1)/2} \frac{(\rho_{11} + \tau_{1j} \tau_{1j})}{\rho_{11}^{\frac{1}{2}}} e^{-\frac{1}{2}r^2} \int_s e^{-\frac{1}{2}} [x_j x_j + \frac{(\tau_{ij} x_j)^2}{\rho_{11}}]\, dx_j \tag{8}$$

where j runs from 2 to n

But if $[\quad] = a_{jk} x_j x_k$, then

$$\int_s e^{-\frac{1}{2}[\quad]}\, dx_j = \frac{(2\pi)^{\frac{n-1}{2}}}{|a_{jk}|^{\frac{1}{2}}}$$

In the case of Eqn (8) $a_{jk} = \delta_{jk} + \frac{\tau_{1j}\tau_{1k}}{\rho_{11}}$

and it is proved in the appendix that for this form

$$|a_{jk}| = 1 + \frac{\tau_{1j}\tau_{1j}}{\rho_{11}}$$

Hence (8) becomes

$$f_s = \frac{1}{2\pi}(\rho_{11} + \tau_{1j}\tau_{1j})^{\frac{1}{2}} e^{-\frac{1}{2}r^2} \tag{9}$$

Now for the boundary in general position $x_i n_i = r$ the frequency will be given by the invariant form which reduces to this as a special case and this is plainly

$$f_s = \frac{1}{2\pi}(\rho_{ij}n_i n_j + \tau_{ij}\tau_{kj}n_i n_k)^{\frac{1}{2}} e^{-\frac{1}{2}r^2} \tag{10}$$

For a spherical region the boundary s is given by:

$$x_i x_i = r^2$$

Since now the normal is parallel to the position vector

$$x_i = r\, n_i$$

all the antisymmetric terms in τ are zero. Hence (7) becomes:

$$f_s = \left(\frac{1}{2\pi}\right)^{(n+1)/2} e^{-r^2/2} \int_s \sigma(\underline{n})\, ds$$

This can be reduced to an integral over the unit sphere S_1 and the coordinates can be rotated so that σ is in principal axes

Hence

$$f_s = \left(\frac{1}{2\pi}\right)^{(n+1)/2} r^{n-1} e^{-\frac{1}{2}r^2} \int_{s_1} \left[\sum \rho_{ii} n_i^{\,2}\right]^{\frac{1}{2}} ds_1 \tag{11}$$

In two-dimensions this is

$$f_s = \frac{r}{(2\pi)^{3/2}} \; e^{-\frac{1}{2}r^2} \int_0^{2\pi} \left[\rho_{11} \cos^2 \theta + \rho_{22} \sin^2 \theta\right]^{\frac{1}{2}} d\theta$$

$$= \frac{r}{(2\pi)^{3/2}} \; e^{-\frac{1}{2}r^2} \; 4 \; \rho_{11}^{\frac{1}{2}} \; E(\alpha) \qquad (12)$$

where $\cos \alpha = \left(\frac{\rho_{22}}{\rho_{11}}\right)^{\frac{1}{2}}$ and $E(\alpha)$ is the complete elliptic integral of the 2nd kind.

ASYMPTOTIC APPROXIMATION

In the general case the integral in (9) would need to be evaluated numerically. However, the main interest will be in cases where crossings out of the safe region occur relatively infrequently. This means, in the way that the problem has been scaled, that the distance, r, of s from the origin is large enough for the term $e^{-\frac{1}{2}r^2}$ to be small. In this case an asymptotic approximation to the integral can be found by considering that the main contribution to it comes just from the neighbourhood of the nearest point to the origin.

For an integral over an (n-1) dimensional space of the form

$$\frac{1}{(2\pi)^{\left(\frac{n-1}{2}\right)}} \int_s e^{-f(x_i)} \; g(x_i) \; ds \qquad (13)$$

let the minimum value of $f(x_i)$ occur at x_i^0. Then expanding $f(x_i)$ in a Taylor series at x_i^0 the first order terms disappear since it is a minimum and the second order terms give an (n-1) dimensional Gaussian integral so that the main contribution to (13) is

$$\frac{g(x_i^0) \; e^{-f(x_i^0)}}{Ho^{\frac{1}{2}}} \qquad (14)$$

where $Ho = \left| \frac{\partial^2 f(x_i^0)}{\partial x_j \partial x_k} \right|$ is the Hessian of f at x^0. For the integral in (7) the minimum occurs at the minimum of r over s since at this point $x_i = rn_i$ so that the contribution from the τ terms is zero and hence also at a minimum.

The axes can be rotated so that the minimum point occurs on the x_1 axis and so that the other axes are in the directions of principal curvature of s.

For simplicity it will be assumed that this is already the case. Then in the neighbourhood of the minimum point s has the form

$$x_1 = r - \tfrac{1}{2}\sum_{i=2}^{n} \kappa_i x_i^2$$

where κ_i are the principal curvatures.

Then the normal in this neighbourhood is given by

$$n_1 = 1,\ n_i = \kappa_i x_i,\ i = 2,\ldots.n$$

Then $$\frac{\partial}{\partial x_i}\left[\frac{\tau_{k1} n_k x_1}{\sigma(\underline{n})}\right]_o = \frac{\tau_{1i}\,(1-\kappa_i r)}{\rho_{11}^{\frac{1}{2}}}$$

Hence

$$\left\{\sigma(\underline{n}) + \tau_{ij} n_i \frac{\partial}{\partial x_j}\left[\frac{\tau_{k1} n_k x_1}{\sigma(\underline{n})}\right]\right\}_o = \frac{1}{\rho_{11}^{\frac{1}{2}}}\left\{\rho_{11} + \sum_{i=2}^{n} (1-\kappa_i r)\,\tau_{1i}^2\right\}$$

Also

$$\frac{\partial^2}{\partial x_i \partial x_j}\tfrac{1}{2}\left[x_k x_k + \frac{(\tau_{k1} n_k x_1)^2}{\sigma^2(\underline{n})}\right]_o = (1-r\kappa_i)\,\delta_{ij} + \frac{\tau_{1i}\tau_{1j}(1-\kappa_i r)(1-\kappa_j r)}{\rho_{11}}$$

Hence using the result proved in the appendix

$$Ho = \frac{1}{\rho_{11}}(1-\kappa_2 r)(1-\kappa_3 r)\ldots..(1-\kappa_n r)\ \left\{\rho_{11} + \sum_{i=1}^{n}(1-\kappa_i r)\tau_{1i}^2\right\}$$

Therefore putting these results together in (14) the asymptotic approximation becomes

$$f_s \sim \frac{1}{2\pi}\left[\frac{\rho_{11} + \sum_{i=2}^{n}(1-\kappa_i r)\tau_{1i}^2}{(1-\kappa_2 r)(1-\kappa_3 r)\ldots..(1-\kappa_n r)}\right]^{\frac{1}{2}} e^{-\frac{1}{2}r^2} \qquad (15)$$

This gives a remarkably simple formula for calculating the outcrossing frequency. Comparison with (9) shows that it becomes exact for a linear space and the extra terms in the denominator show how the increased nearness of contact of the surface with the sphere about the origin due to the surface curvature increases the result. The solution becomes infinite when one of these terms is zero and this occurs when the contact with the sphere is of a higher order than simple tangential touching. In that case the approximation breaks down and needs to be taken to higher order than the second giving rise to Airy functions in the simplest case. These cases will not be followed through here, since the possibilities are so various as to over complicate a general treatment.

EXAMPLES

The theory will now be illustrated with some examples. As these will all be two-dimensional, the formulae already given can be restricted to that case.

In two variables the only non-zero value of the cross-correlation is τ_{12} which will be written simply as τ. Also $n_1 \frac{\partial}{\partial x_2} - n_2 \frac{\partial}{\partial x_1}$ becomes simply $\frac{d}{ds}$ where s is arc length along the boundary curve.

Hence (7) becomes

$$f_s = \frac{1}{(2\pi)^{3/2}} \int_s e^{-\frac{1}{2}\left[r^2 + \frac{\tau^2}{\sigma^2}(n_1x_2 - n_2x_1)^2\right]} \left\{\sigma + \tau^2 \frac{d}{ds}\left[\frac{n_1x_2 - n_2x_1}{\sigma}\right]\right\} ds \qquad (16)$$

where $\sigma^2 = \rho_{11} n_1^2 + \rho_{22} n_2^2 + 2\rho_{12} n_1 n_2$

The asymptotic approximation (15) becomes

$$f_s \sim \frac{1}{2\pi} \left[\frac{\rho_{11}}{1-\kappa r} + \tau^2\right]^{\frac{1}{2}} e^{-\frac{1}{2}r^2}$$

but since τ^2 is independent of the orientation of the coordinates and ρ_{11} is just the value of σ^2 at the particular special position for which (15) was derived, this can be written for general position of the minimum as

$$f_s \sim \frac{1}{2\pi} \left[\frac{\sigma^2(\underline{n})}{1-\kappa r} + \tau^2\right]^{\frac{1}{2}} e^{-\frac{1}{2}r^2}$$

where $\underline{n}$ is the normal at the minimum.

Amplitude of vectors

The first example will be the case of the extremes of the amplitude of two-dimensional vectors treated in Ref 2.

Let the vectors be (Y_1, Y_2) where the components have the spectra $S_{11}(\omega)$, $S_{22}(\omega)$ and the cross spectrum $S_{12}(\omega)$ and the problem is to determine the out crossing frequency of the circle $Y_1^2 + Y_2^2 = R^2$.

Define the moments of the spectra:

$$m_n = \int_0^\infty \omega^n S_{11}(\omega)\, d\omega \; ; \; m_n' = \int_0^\infty \omega^n S_{22}(\omega)\, d\omega$$

$$C_n + iq_n = \int_0^\infty \omega^n S_{12}(\omega)\, d\omega$$

Then $\{Y_1^2\} = m_0 \qquad \{Y_2^2\} = m_0' \qquad \{Y_1 Y_2\} = c_0$

$\{\dot{Y}_1^2\} = m_2 \qquad \{\dot{Y}_2^2\} = m_2' \qquad \{\dot{Y}_1\dot{Y}_2\} = c_2$

$\{Y_1\dot{Y}_2\} = -q_1$

It will be assumed that $m_0 > m_0'$ and that the displacements are already in principal axes so that $c_0 = 0$. Then they are reduced to a unit isotropic form by putting

$$x_1 = \frac{Y_1}{m_0^{\frac{1}{2}}} \qquad x_2 = \frac{Y_2}{m_0'^{\frac{1}{2}}}$$

Then $\lambda_{11} = \dfrac{m_2}{m_0} \qquad \lambda_{22} = \dfrac{m_2'}{m_0'} \qquad \lambda_{12} = \dfrac{c_2}{(m_0 m_0')^{\frac{1}{2}}}$

$$\tau = -\frac{q_1}{(m_0 m_0')^{\frac{1}{2}}}$$

and from (3) $\rho_{11} = \lambda_{11} - \tau^2$; $\rho_{22} = \lambda_{22} - \tau^2$; $\rho_{12} = \lambda_{12}$. The boundary becomes the ellipse

$$\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} = 1$$

where $a_1 = \frac{R}{m_0^{\frac{1}{2}}}$; $a_2 = \frac{R}{m_0'^{\frac{1}{2}}}$ so that $a_2 > a_1$.

Paramaterize the ellipse by putting $x_1 = a_1 \cos\theta$; $x_2 = a_2 \sin\theta$.

Then $$n_1 = \frac{a_2 \cos\theta}{(a_1^2 \sin^2\theta + a_2^2 \cos^2\theta)^{\frac{1}{2}}}$$

$$n_2 = \frac{a_1 \sin\theta}{(a_1^2 \sin^2\theta + a_2^2 \cos^2\theta)^{\frac{1}{2}}}$$

$$\frac{ds}{d\theta} = (a_1^2 \sin^2\theta + a_2^2 \cos^2\theta)^{\frac{1}{2}}$$

Hence from (16) the outcrossing frequency is given by the integral

$$f_s = \frac{1}{(2\pi)^{3/2}} \int_o^{2\pi} e^{-\frac{1}{2}\left[a_1^2\cos^2\theta + a_2^2 \sin^2\theta + \tau^2 h^2(\theta)\right]} \left\{g(\theta) + \tau^2 \frac{dh(\theta)}{d\theta}\right\} d\theta$$

where $g(\theta) = \left[\rho_{11}\, a_2^2 \cos^2\theta + \rho_{22}\, a_1^2 \sin^2\theta + 2\rho_{12}\, a_1 a_2 \sin\theta \cos\theta\right]^{\frac{1}{2}}$

and $$h(\theta) = \frac{(a_2^2 - a_1^2) \sin\theta \cos\theta}{g(\theta)}$$

If $a_2 = a_1$ then this simplifies to the result given for a circular region.

If $a_2 > a_1$ then there are two equal minima on the x_1 axis. In the neighbourhood of the point (a_1, o)

$$x_1 = a_1 - \frac{1}{2} \frac{a_1}{a_2^2} x_2^2$$

hence $\kappa = \frac{a_1}{a_2^2}$ and $\kappa r = \frac{a_1^2}{a_2^2}$

Therefore the asymptotic approximation (17) gives when doubled for the second minimum:

$$f_s \sim \frac{1}{\pi}\left[\frac{\rho_{11}}{1-\frac{a_1^2}{a_2^2}} + \tau^2\right]^{\frac{1}{2}} e^{-\frac{1}{2}a_1}$$

or in terms of the original variables

$$f_s \sim \frac{1}{\pi}\left[\frac{m_2 - q_1{}^2/m_0}{m_0 - m_0'}\right]^{\frac{1}{2}} e^{-\frac{1}{2}R^2/m_0}$$

<u>Wave paddle end stop</u>

A controlled wave paddle generating a random sea will have some form of end stop device to prevent damage. We wish to calculate the probability of hitting the stop and thereby limiting the paddle performance. If the paddle has displacement y with spectrum $S(\omega)$ and the moments of the spectrum are defined by:

$$m_n = \int_0^\infty \omega^n S(\omega)\, d\omega$$

Then for a stop at $y = R$, the mean frequency of hits is given by:

$$f_s = \frac{1}{2\pi}\left(\frac{m_2}{m_0}\right) e^{-\frac{1}{2} R^2/m_0} \qquad (18)$$

Now suppose the paddle has a more complicated device which also depends on its velocity and acts to allow the paddle to decelerate at a constant rate to come to rest at $y = R$. Then the boundary in the $(Y,\dot{Y})$ plane crossing which activates the stop is given by

$$\dot{Y}^2 = 2a\,(R-Y)$$

where a = the rate of deceleration.

To reduce to our standard form put

$$x_1 = \frac{Y}{m_0^{\frac{1}{2}}} \qquad x_2 = \frac{\dot{Y}}{m_2^{\frac{1}{2}}}$$

Then $\lambda_{11} = \frac{m_2}{m_0}$; $\lambda_{22} = \frac{m_4}{m_2}$; $\lambda_{12} = 0$; $\tau = -\left(\frac{m_2}{m_0}\right)^{\frac{1}{2}}$

Hence $\rho_{11} = 0$; $\rho_{22} = \frac{m_4 m_0 - m_2^{\,2}}{m_2 m_0}$; $\rho_{12} = 0$

Parameterize the boundary by putting

$$x_1 = \alpha\,(\beta - \theta^2)$$

$$x_2 = 2\,\alpha\theta$$

where $\alpha = \frac{a m_0^{\frac{1}{2}}}{2m_2}$ and $\beta = \frac{2m_2 R}{m_0 a}$

Then $\frac{ds}{d\theta} = 2\alpha\,(1 + \theta^2)^{\frac{1}{2}}$

$n_1 = \frac{1}{(1+\theta^2)^{\frac{1}{2}}}$; $n_2 = \frac{\theta}{(1+\theta^2)^{\frac{1}{2}}}$

and $r^2 = \alpha^2\left[(\beta-\theta^2)^2 + 4\theta^2\right]$; $\sigma^2 = \rho_{22}\frac{\theta^2}{1+\theta^2}$

Whence (16) becomes

$$f_s = \frac{2\alpha\;\rho_{22}^{\frac{1}{2}}\left(1+\frac{\tau^2}{\rho_{22}}\right)}{(2\pi)^{3/2}}\int_0^\infty e^{-\frac{\alpha^2}{2}\left[(\beta-\theta^2)^2 + 4\theta^2 + \frac{\tau^2}{\rho_{22}}(\theta^2 + 2 - \beta)^2\right]}\,\theta d\theta$$

There are two stationary values of r, one at $\theta = 0$ and one at $\theta = (\beta-2)^{\frac{1}{2}}$.

If $\beta < 2$, that is if the deceleration is sufficiently large, then the minimum occurs at $\theta = 0$ and the asymptotic approximation is identical to the simple end stop (18). If $\beta > 2$ then the other value gives the minimum.

Here $r = 2\alpha\,(\beta-1)^{\frac{1}{2}}$; $\kappa r = \frac{1}{(\beta-1)}$; $\sigma^2 = \rho_{22}\frac{\beta-2}{\beta-1}$

so that (17) gives

$$f_s \sim \frac{1}{2\pi}\,(\rho_{22} + \tau^2)^{\frac{1}{2}}\,e^{-2\alpha^2(\beta-1)}$$

$$= \frac{1}{2\pi}\left(\frac{m_4}{m_2}\right)^{\frac{1}{2}} e^{-\frac{1}{2}\frac{a^2 m_0}{m_2^{\,2}}\left(\frac{2m_2 R}{m_0 a} - 1\right)}$$

Morison's Equation

The force on an object in an accelerating flow is commonly approximated by Morison's equation as the sum of inertial and drag components. For a time dependent velocity, u, the force is of the form

$$F = A\dot{u} + B\,u|u|$$

To find the mean rate of exceedance of the force, put

$$X_1 = \frac{\dot{u}}{m_2^{\frac{1}{2}}} \quad ; \quad X_2 = \frac{u}{m_0^{\frac{1}{2}}}$$

where the m_n are the moments of the spectrum of u.

Then $\lambda_{11} = \dfrac{m_4}{m_2}$; $\lambda_{22} = \dfrac{m_2}{m_0}$; $\lambda_{12} = 0$; $\tau = \left(\dfrac{m_2}{m_0}\right)^{\frac{1}{2}}$

so that $\rho_{11} = \dfrac{m_4 m_0 - m_2^2}{m_2 m_0}$; $\rho_{22} = \rho_{12} = 0.$

The boundary is parameterized by

$$X_1 = \alpha(\beta - \theta|\theta|) \ ; \ X_2 = 2\alpha\theta$$

where $\alpha = \dfrac{A m_2^{\frac{1}{2}}}{4Bm_0}$; $\beta = \dfrac{4Bm_0 F}{A^2 m_2}$

Thus the geometry is very similar to the previous example but the axes of the variance are changed over so that now

$$\sigma^2 = \rho_{11} \frac{1}{1+\theta^2}$$

This apparently slight change means that the mean frequency (16) now becomes the more complicated:

$$f_s = \frac{2\alpha\, \rho_{11}^{\frac{1}{2}}}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} e^{-\frac{\alpha^2}{2}\left[(\beta-\theta|\theta|) + 4\theta^2 + \frac{\tau^2}{\rho_{11}}(\theta^3 + 2\theta - \beta|\theta|)^2\right]}$$

$$\left\{1 + \frac{\tau^2}{2\rho_{11}}(3\theta^2 + 2 - \beta \operatorname{sgn} \theta)\right\} d\theta$$

The stationary values of r are the same as before. For $\beta > 2$, that is for sufficiently large force values the minimum is at $\theta = (\beta-2)^{\frac{1}{2}}$. The radius and curvature are the same as before but $\sigma^2 = \rho_{11} \frac{1}{\beta-1}$ so that (17) gives

$$f_s \sim \frac{1}{2\pi}\left(\frac{\rho_{11}}{\beta-2} + \tau^2\right)^{\frac{1}{2}} e^{-2\alpha^2(\beta-1)} \qquad (19)$$

Thus in this case the approximation breaks down as $\beta \to 2$ and the two stationary values coincide.

For $\beta < 2$, the minimum occurs at $\theta = 0$, hence taking the asymptotic approximation in two halves depending on the sign of θ

$$f_s \sim \frac{1}{4\pi}\left[\left(\frac{\rho_{11}}{1-\beta/2} + \tau^2\right)^{\frac{1}{2}} + \left(\frac{\rho_{11}}{1+\beta/2} + \tau^2\right)^{\frac{1}{2}}\right] e^{-\frac{1}{2}\alpha^2\beta^2} \qquad (20)$$

For $\beta = 2$, the region $\theta > 0$ can be taken to the next order of approximation giving

$$f_s \sim \frac{1}{4\pi}\left[\frac{4\alpha\, \rho_{11}^{\frac{1}{2}}}{(2\pi)^{\frac{1}{2}}} \int_0^{\infty} e^{-\frac{\alpha^2\theta^4}{2}} d\theta + \left(\frac{\rho_{11}}{2} + \tau^2\right)^{\frac{1}{2}}\right] e^{-2\alpha^2}$$

$$= \frac{1}{4\pi}\left[\left(\frac{\alpha\, \rho_{11}}{2\pi}\right)^{\frac{1}{2}} 2^{\frac{1}{4}}\, \Gamma\left(\tfrac{1}{4}\right) + \left(\frac{\rho_{11}}{2} + \tau^2\right)^{\frac{1}{2}}\right] e^{-2\alpha^2}$$

Taking the drag force alone would give

$$f_s = \frac{1}{2\pi}\left(\frac{m_2}{m_0}\right)^{\frac{1}{2}} e^{-\frac{1}{2}\frac{F}{Bm_0}} = \frac{1}{2\pi}\, \tau\, e^{-2\alpha^2\beta}$$

which corresponds to (19) for large β

Taking the inertial force alone would give

$$f_s = \frac{1}{2\pi} \left(\frac{m_4}{m_2}\right)^{\frac{1}{2}} e^{-\frac{1}{2} \frac{F^2}{A^2 m_2}} = \frac{1}{2\pi} (\rho_{11} + \tau^2)^{\frac{1}{2}} e^{-\frac{1}{2} \alpha^2 \beta^2}$$

which corresponds to (20) for small β.

This treatment can easily be extended to include a steady current in the drag force. The variables are defined as before and the boundary shape remains the same; it is merely displaced in the x_2 direction by the current.

CONCLUSIONS

A general method has been developed which can be used to estimate the extremes of a parameter which is defined by an explicit combination of quantities which themselves are derivable by linear transformation from a common random excitation (the sea state). The method gives extreme time domain statistics, the input to the problem being frequency domain information such as spectral and cross-spectral moments between the various components quantities.

Examples have been demonstrated which cover:

(i) estimation of the extremes of the resultant of two orthogonal gaussian components,

(ii) a hydraulic actuator which has end stop conditions defined in terms of both displacement and velocity

(iii) estimation of the extreme force on a structure using the Morison method of description in a random sea state.

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APPENDIX

Denote by Δ_n the determinant of an nth order matrix of the form

$$\delta_{ij} + c_i d_j$$

Then separating off the term arising from δ_{nn}

$$\Delta_n = \Delta_{n-1} + Z_n$$

where Z_n is the same determinant with $\delta_{nn} = 0$

Then subtracting $\frac{c_i}{c_n}$ (nth row) from ith row i = 1,2

...n-1, shows that $Z_n = c_n d_n$

Hence since $\Delta_1 = 1 + c_1 d_1$

$$\Delta_n = 1 + c_1 d_1 + c_2 d_2 \cdots\cdots + c_n d_n$$

Then plainly for a matrix of the form

$$b_i \, (\delta_{ij} + c_i d_j)$$

$$\Delta_n = b_1 b_2 \cdots b_n \, (1 + c_1 d_1 + c_2 d_2 \cdots\cdots + c_n d_n)$$