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THE DEVELOPMENT OF A FINITE DIFFERENCE MODEL
FOR THE PROPAGATION OF STORM WAVES IN SHALLOW
WATERS

by

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ABSTRACT

A finite difference model has been developed for the simulation of storm waves propagating in coastal waters. These waves usually have periods from 5 to 20 secs and if their wave length is still large in comparison with the water depth, then their motion is described by the so-called Boussinesq equations. These are basically the well known shallow water ("long wave") equations but with additional terms in the momentum equation to account for non-hydrostatic pressure and frequency dispersion. For the numerical integration of these terms, we introduce an iterative procedure of the predictor-corrector type.

The model is validated against simple analytical solutions both in one and two dimensions and, also against a few physical model data. It is found to simulate successfully all non-linear and dispersive terms in the equations and also reflection and diffraction effects, wave absorbing boundaries and set-down beneath wave groups. All tests have been carried out for a constant depth and therefore, wave refraction effects have been excluded.

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1 INTRODUCTION

Civil engineering practice is greatly concerned with the important effects of the so-called "short waves" as they propagate in shallow coastal waters. These are storm waves having periods from 5 to 25 secs which from the hydrodynamic point of view may be considered in certain cases as long waves, if their wave length remains large in comparison with the water depth. Their multiple action constitutes an agitating factor of substantial power in the vicinity of coastal harbours in a number of ways:

- they can attack directly defensive or other on-shore structures (e.g. breakwaters, piers, sea walls) causing damage;
- they can penetrate through harbour entrances to disturb the waters within and set in motion by their direct or combined action moored ships, thus resulting in delays in the port operations;
- they can bring sediment into suspension causing coastal erosion;
- they induce currents due to their non-linearities and eventual breaking which, in their turn, transport suspended sediments in other regions as littoral drift; in this way, they may affect adversely navigational or possibly dredging operations. Therefore there is little doubt why there is much practical interest in this sort of waves which has led over the years to a considerable effort worldwide for the development of basic techniques to predict their behaviour either along the coast or in and around harbours and engineering works. Even though most of today's solving methods rely largely on physical modelling, a substantial amount of work has

been devoted, during the last ten years or so, to the development of computational techniques for the simulation, particularly, of their non-linear propagation over relatively shallow water depths. From the mathematical point of view, this is fundamentally described by the well-known shallow water ("long wave") equations of nearly horizontal flow which are obtained by integrating the basic hydrodynamic equations of conservation of mass and momentum over the mean depth und certain assumptions:

Conservation of mass:

$$\frac{\partial z}{\partial t} + \frac{\partial}{\partial x} (u.h) + \frac{\partial}{\partial y} (v.h) = 0 \quad (1)$$

Conservation of momentum:

$$\begin{aligned} \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} &= -g \frac{\partial z}{\partial x} \\ \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} &= -g \frac{\partial z}{\partial y} \end{aligned}$$

where

z elevation above datum

u,v depth averaged components of velocity in x,y direction respectively

h total depth = d + z, d mean water depth

The fluid is taken to be homogenous, isotropic and incompressible, while all shear stress and Coriolis effects have been neglected. These equations are extensively used in river and tidal hydraulics describing thereby the water motion with considerable accuracy. Nevertheless, they are derived by assuming the vertical accelerations to be negligible (i.e. nearly horizontal flow) and such an assumption leads to hydrostatic pressure within the fluid, while the

waves are not dispersive travelling with a speed depending only on the water depth. This is realistic only for very long waves such as tides.

Boussinesq (Ref 16) was the first to deviate from such an assumption by introducing a linear variation of the vertical velocity w from the bottom to the surface and by doing this he succeeded in incorporating higher order terms in the equations representing the effect of small, but not negligible, vertical accelerations due to the curvature of the streamlines. The pressure is no longer hydrostatic and the waves retain their dispersive character i.e. their speed depends on both their wavelength and the water depth. For rather steep, non-linear waves this speed depends on wave height as well.

In recent years, Peregrine (Ref 11) extended Boussinesq's ideas to two spatial dimensions and he also considered not a constant but a gently varying sea bed. He presented the equations in the form by which they are now known, i.e:

Conservation of mass:

$$\frac{\partial z}{\partial t} + \frac{\partial}{\partial x} (u.h) + \frac{\partial}{\partial y} (v.h) = 0 \quad (\text{i.e. same as before})$$

Conservation of momentum:

$$\begin{aligned} \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = & -g \frac{\partial z}{\partial x} + \frac{1}{2} h \left[\frac{\partial^3 (u.h)}{\partial x^2 \partial t} \right. \\ & \left. + \frac{\partial^3 (v.h)}{\partial x \partial y \partial t} \right] - \frac{1}{6} h^2 \left[\frac{\partial^3 u}{\partial x^2 \partial t} + \frac{\partial^3 v}{\partial x \partial y \partial t} \right] \quad (2) \end{aligned}$$

$$\begin{aligned} \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = & -g \frac{\partial z}{\partial y} + \frac{1}{2} h \left[\frac{\partial^3 (u.h)}{\partial x \partial y \partial t} \right. \\ & \left. + \frac{\partial^3 (v.h)}{\partial y^2 \partial t} \right] - \frac{1}{6} h^2 \left[\frac{\partial^3 u}{\partial x \partial y \partial t} + \frac{\partial^3 v}{\partial y^2 \partial t} \right] \end{aligned}$$

The inclusion of third order terms in the equations poses particular problems in their numerical integration and care must be taken the numerical error involved is not comparable with these terms, otherwise it would contaminate their proper effect. Appropriate techniques have been invented first by Abbott et al (Ref 1), then by Hauguel (Ref 8) and recently by Rottmann-Söde, Schaper and Zielke (Ref 13) using implicit finite difference schemes.

The present report describes similar work, even though it follows a slightly different approach regarding the numerical modelling of the Boussinesq equations. It adopts an iterative procedure of the predictor-corrector kind for the approximation of the higher order terms. This is done with a view to taking best advantage of the existing high power computer capabilities at HR by means of a DAP (Distributed Array Processor) system of parallel processing. This is particularly important for the modelling of "short" wave propagation at low operational costs, since computational accuracy depends on small Courant numbers (≈ 1) and a fair number of grid points per wave length and time steps per wave period. The report gives details on the fundamental mathematical problem, the development of a numerical model for its solution, the validation of this model against a number of exact analytical solutions and physical model data and the assumed absorbing boundary conditions. The main assumptions involved and the range of application of the finite difference (FD) scheme are clearly stated.

The model is also applied to study the diffraction around a single breakwater arm of the so-called set-down wave. This is the result of non-linear interactions between short, primary waves (Ref 3) and it has great practical significance in a harbour design: its long period (approx 30 secs to a few minutes) lies within the range of natural periods of the horizontal oscillations of a moored ship and,

therefore, its presence in a harbour may lead to resonant conditions and, eventually, to high loads on the mooring lines.

All experiments described in this report have been carried out assuming a constant depth, therefore, further work will be needed to include a variable sea bottom and, thus, all shoaling and refraction effects within the model. Similarly, the performance of the model should be further investigated in case the input conditions are provided by means of random waves corresponding to a certain incident wave energy spectrum (in terms of frequency or both in frequency and direction).

2 BACKGROUND INFORMATION

2.1 Brief description of wave theories

Deep water waves of small height have an almost sinusoidal profile and thus they are best described mathematically by the linear Airy theory. This theory assumes the wave height to be small in comparison with both the water depth and the wave length (i.e. $H/d \ll 1$, $\frac{H}{L} \ll 1$) but it imposes no restriction on the water depth/wave length ratio d/L . The latter can be used to identify three distinct areas as follows:

$$\frac{d}{L} < 1/2 \text{ shallow waters}$$

$$\frac{1}{20} < \frac{d}{L} < 1/2 \text{ waters of intermediate depth}$$

$$\frac{d}{L} > 1/2 \text{ deep waters}$$

In the last two regions, the waves are dispersive, that is their speed depends on their wave length, and the pressure is non-hydrostatic. Nevertheless, in shallower waters, the waves become non-dispersive,

since their speed starts to depend only on the water depth and the pressure is only hydrostatic. As the depth decreases towards the coast, their speed reduces continuously and the waves are retarded. The result is a continuous transformation of their kinetic to potential energy and thus, a continuous building up of their amplitude. The assumptions of the linear theory are very soon invalidated. Even though the linear approach has been found to be realistic, even when there is quite a major departure from the small wave height assumption nevertheless it is best to use Stokes wave theory in order to describe waves of finite amplitude in shallow waters. Stokes (Ref 14) introduced a perturbation method in which the variables of the flow are developed as power series in terms of a small perturbation parameter. It is expected that the various series will converge as the number of terms considered increases. In practice, the 5th order Stokes theory has found wide application, particularly in wave force calculations.

Again, if we consider a shallower range of depths, Stokes' theory becomes inadequate, since in such a case many coefficients of the higher order terms "blow up" meaning that they become excessive relative to the lowest order terms. The Stoke's wave expansion method is formally valid when $\frac{H}{d} \ll (kd)^2$, for $kd \ll 1$ and $\frac{H}{L} \ll 1$.

The above conditions place a severe wave height restriction in shallow waters and, therefore, a different approach is needed. In this respect, the earliest relevant theory is that of Airy (Ref 1, 16) who makes the assumption that, since the wave length is much larger than the water depth, the vertical particle accelerations may be neglected and thus, the pressure is hydrostatic. Airy's theory is noted for implying that waves cannot propagate over a horizontal

bottom without a change of their profile; their speed depends on their height and, therefore, different parts of the wave travel with different speed. Consequently, they will eventually break. If the wave height is rather small, then we can neglect all non-linear terms in the governing equations and this case is covered by the linearised shallow water, "long wave" theory by Jeffreys & Jeffreys (Ref 1) which again assumes hydrostatic pressure. This theory has found wide application in tidal motion, tsunami propagation, storm surge, flood waves and the like.

Between the theories of Airy and Jeffreys and Jeffreys, there lies the theory of Boussinesq (Ref 1). According to this theory, the vertical particle velocity varies linearly from zero at the sea bottom to a maximum at the free surface and therefore the vertical acceleration is still small but not negligible. The pressure is no longer hydrostatic but the vertical component of motion can be integrated out of the equations to reduce the three-dimensional description to a two-dimensional one. The original Boussinesq equations held only for horizontal bottom and uni-directional propagation (see Eq 6). Peregrine recently derived the full Boussinesq-type equations governing the propagation of arbitrary, long wave disturbances of small to moderate amplitude over a slowly varying bathymetry (see Eq 2).

Finally, Korteweg and de Vries (Ref 1, 16) working along parallel lines modified the method of Boussinesq and presented a second theory of waves moving in shallow waters. According to their theory, the wave characteristics are expressed in terms of the Jacobian elliptic function cn and hence, the terminology "cnoidal wave theory".

A typical cnoidal wave profile is shown in Fig 1(a). One limiting case of this, in which the wave length becomes infinite, corresponds to the solitary wave,

while another one corresponds to shallow water sinusoidal wave theory (see Fig 1(b),(c)). The different behaviour of the waves is best described by the Ursell parameter:

$$U_r = \frac{a/d}{\left(\frac{d}{L}\right)^2} = \frac{a \cdot L^2}{d^3} \quad (3)$$

where a the wave amplitude.

The ratio a/d controls the effect of the non-linear terms in the equations while the ratio $(d/L)^2$ controls that of the dispersive terms. For large values of U_r (i.e. $U_r \gg 1$) the non-linear effects completely dominate the flow and the wave motion is described by the Airy equations. For small values of U_r (i.e. $U_r \ll 1$), the theory of linear long waves by Jeffreys and Jeffreys is applied. If $U_r = O(1)$, then both non-linear and dispersive effects are important and the motion is described by the Boussinesq-type equations (see Eq 2, 6). One special case of these equations for waves moving in one direction from left to right, is the KdV (Korteweg de Vries) type equation given below (see Ref 16).

$$\frac{\partial z}{\partial t} + \sqrt{gd} \left(1 + \frac{3}{2} \frac{z}{d}\right) \cdot \frac{\partial z}{\partial x} + \frac{1}{6} \sqrt{gd} d^2 \frac{\partial^3 z}{\partial x^3} = 0 \quad (4)$$

z , d as before.

This equation has permanent cnoidal and solitary waves as solutions.

In recent years, both the Boussinesq and KdV type equations have been further extended:

- (a) Serre (Ref 2, 8) modified the Boussinesq equations by adding higher order terms, products of derivatives which Boussinesq had neglected. They take the form:

$$\frac{\partial h}{\partial t} + \frac{\partial p}{\partial x} + \frac{\partial q}{\partial t} = 0$$

$$\begin{aligned} & \frac{\partial p}{\partial t} + \frac{\partial}{\partial x} \left(\frac{p^2}{h} \right) + \frac{\partial}{\partial y} \left(\frac{p \cdot q}{h} \right) + \frac{\partial}{\partial x} \left(\left(\frac{g+\beta}{2} + \frac{\alpha}{3} \right) \cdot h^2 \right) \\ & = - \left(g + \beta + \frac{\alpha}{2} \right) \cdot h \frac{\partial Z}{\partial x} - g \frac{p}{C^2} \frac{\sqrt{p^2 + q^2}}{h^2} \end{aligned} \quad (5)$$

$$\begin{aligned} & \frac{\partial q}{\partial t} + \frac{\partial}{\partial x} \left(\frac{p \cdot q}{h} \right) + \frac{\partial}{\partial y} \left(\frac{q^2}{h} \right) + \frac{\partial}{\partial y} \left(\left(\frac{g+\beta}{2} + \frac{\alpha}{3} \right) \cdot h^2 \right) \\ & = - \left(g + \beta + \frac{\alpha}{2} \right) \cdot h \frac{\partial Z}{\partial y} - g \cdot \frac{q}{C^2} \frac{\sqrt{p^2 + q^2}}{h^2} \end{aligned}$$

where:

h total water depth
p, q x- and y- volume fluxes
Z bed elevation
C friction coefficient

$$\alpha = \frac{d^2 h}{dt^2}, \quad \beta = \frac{d^2 Z}{dt^2}$$

$$\text{with } \frac{d}{dt} = \frac{\partial}{\partial t} + \frac{p}{h} \frac{\partial}{\partial x} + \frac{q}{h} \frac{\partial}{\partial y}$$

Hauguel (Ref 8) and Abbott et al (Ref 2) presented numerical solutions of these equations.

- (b) In so far as the KdV equation is concerned, presentations of a first approximation to this theory have been made by Keulegan and Patterson, Keller and Laitone (Ref 14). Laitone and Chappellear (Ref 14) have developed respectively 2nd and 3rd order approximations to cnoidal wave theory. More recently, Fenton (Ref 14) has presented a cnoidal wave theory which is capable of extension to any desired order and which is readily suited to engineering application. Also, Friedrichs (Ref 14) following a different

approach proposed a rigorous perturbation method for developing shallow water theories by means of the parameter $\sigma = \left(\frac{d}{L}\right)^2 \ll 1$. His expansion of the variables as power series of σ is based on the assumption that $H/d = O[(kd)^2]$ (i.e. that it is of the same order as σ) and he obtained to the lowest order the linearised long wave theory. Keller used the same method to find the KdV equation as a first approximation and Laitone obtained the 2nd order cnoidal solution by the same method (see Ref 14).

2.2 Dispersive properties of waves

Water waves are distinguished in two main classes, hyperbolic and dispersive waves. The first class bears its name because the water motion is mathematically formulated by means of hyperbolic differential equations. The second class owes the name to the type of existing solution rather than the type of equation. If this solution determines the frequency ω as a definite real function of the wave number k i.e. $\omega(k)$, then if the phase speed $\omega(k)/k$ is not the same for all k , the modes with different k will propagate at different speeds, that is, they will "disperse". Nevertheless, it must be noted that the above-mentioned classes are not exclusive: there is some overlap in that certain wave motions exhibit both types of behaviour and there are certain exceptions that fit neither. Within this context, Boussinesq was the first to formulate equations trying to combine both dispersive and hyperbolic effects in the (x,t) domain:

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x} (uh) = 0 \quad (6)$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + g \frac{\partial h}{\partial x} + \frac{1}{3} d \frac{\partial^3 h}{\partial x \partial t^2} = 0$$

by adding the dispersive 3rd order term $\frac{\partial^3 h}{\partial x \partial t^2}$ in the basic shallow water equations.

By considering an inviscid, incompressible and irrotational fluid of constant depth, the basic hydrodynamic equations lead, eventually, to the formulation of the Laplace equation $\nabla^2 \phi = 0$ where ϕ is the flow potential. If we linearise the free surface conditions, then, it is shown that the following dispersion relation holds true:

$$\omega^2 = gk \tanh(kd)$$

This can be approximated in polynomial form as:

$$\omega^2 = gk. (kd - 1/3 d^3 k^3 + \dots) \quad (7)$$

For shallow waters i.e. as $kd \rightarrow 0$, the dispersion relationship above becomes $\omega^2 = gk^2 d$ and the phase speed C_0 becomes $\omega/k = \sqrt{gd}$ i.e. independent of k .

Similarly, the linearised version of the Boussinesq equations above leads to the dispersion relation:

$$\omega^2 = \frac{C_0^2 k^2}{1 + \frac{1}{3} k^2 d} \quad (8)$$

which for small $(kd)^2$ agrees with the expression (7) above to the first two terms. Boussinesq's equations include waves moving to both left and right. By restricting to waves moving to the right only and eliminating the velocity u , we can transform the Boussinesq equations to the equation:

$$\partial z / \partial t + 3(\sqrt{g(d+z)} - 2\sqrt{gd}) \frac{\partial z}{\partial x} + \gamma \frac{\partial^3 z}{\partial x^3} = 0 \quad (9)$$

where $\gamma = \frac{1}{6} C_0 d^2$. This equation combines the non-linear effects of the basic shallow water equations together with dispersive effects presented by the term $\gamma \frac{\partial^3 z}{\partial x^3}$. If the non-linear terms are approximated to the 1st order in a/d (where a is a typical wave amplitude), we obtain the KdV equation (4) (see Ref 16). The interesting thing about this equation is that it predicts permanent waves as solutions by means of the Jacobian elliptic function cn , the so-called cnoidal waves which were mentioned before. As the wave length tends to infinity, a particular solution is the solitary wave, first discovered experimentally by Scott Russell (1844) and analysed mathematically by Boussinesq (1871).

The linear version of this equation leads to the dispersion relation

$$\omega = \frac{C_0 \cdot k}{1 + \gamma \cdot k^2 / C_0} \quad (11)$$

which for small k agrees with the dispersion relation defined by the original set of Boussinesq equations. Thus, it becomes evident that under the circumstances, solutions of the KdV equation (such as cnoidal and solitary waves) should be, also, solutions of the Boussinesq equations for unidirectional flow in a constant water depth d . One of the basic properties of these equations is their ability to transform themselves to different forms by means of the third order term and the linearised long wave equations $z_t = -d \cdot u_x$ and $u_t = -g \cdot h_x$. So, the term $\partial^3 h / \partial x \cdot \partial t^2$ in Eq 6 above can be transformed to $\frac{\partial^3 u}{\partial x^2 \cdot \partial t}$ - a version which leads to the same dispersion relation - and,

also, to $\frac{\partial^3 h}{\partial x^3}$ which via the linearised version of the Boussinesq equations leads to the dispersion relation:

$$\omega^2 = Co^2 \cdot k^2 \left(1 - \frac{1}{3} k^2 \cdot d^2\right). \quad (12)$$

According to this, small perturbations with $(kd)^2 > 3$ will in fact amplify, because ω becomes imaginary whereas dispersion relation (8) above retains only real values of ω . In numerical work, the effects of finite differencing and truncation errors introduce small oscillations of small wave length which may suffer amplification for the above reasons. This is why the version of $\partial h^3 / \partial x \cdot \partial t^2$ (or $\frac{\partial^3 h}{\partial x^2 \cdot \partial t}$) has been eventually selected, i.e. the Boussinesq type equations become:

$$\begin{aligned} \frac{\partial h}{\partial t} + \frac{\partial}{\partial x} (uh) &= 0 \\ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + g \frac{\partial h}{\partial x} - \frac{1}{3} d^2 \frac{\partial^3 h}{\partial x^2 \cdot \partial t} &= 0 \end{aligned} \quad (13)$$

3 ONE DIMENSIONAL SOLUTION

3.1 Linear conditions

We seek a numerical solution of the differential equations (13). These are fundamentally the non-linear shallow water equations which have been found extensive use in tidal and river hydraulics in recent years. For this reason, we use as a basis for their solution the well established finite difference model for tidal flows in the Tidal Engineering Department of HRL (Ref 17). This model uses a grid staggered both in space and time and a leap frog

approximation of the partial derivatives by finite differences. It can be described as a 2nd order accurate, conditionally unstable scheme of the explicit type. Under linear conditions, it takes the form: (see also Fig 2)

$$z_i^{n+1/2} = z_i^{n-1/2} - d \cdot \Delta t \cdot \frac{u_{i+1/2}^n - u_{i-1/2}^n}{\Delta x} \quad (14)$$

$$u_i^{n+1} = u_i^{n-1} - g \cdot \Delta t \cdot \frac{z_{i+1/2}^{n+1/2} - z_{i-1/2}^{n+1/2}}{\Delta x} + \frac{1}{3} d \left[\frac{\partial^3 u}{\partial x^2 \partial t} \right]^{n+1/2} \Delta t$$

where z is the elevation, u the x-velocity and $\left[\frac{\partial^3 u}{\partial x^2 \partial t} \right]$ stands for a certain FD approximation of the term $\partial^3 u / \partial x^2 \partial t$. This approximation can simply be:

$$\left[\frac{\partial^3 u}{\partial x^2 \partial t} \right] = \frac{u_{i+1}^{n+1/2} + u_{i-1}^{n+1/2} - 2u_i^{n+1/2} - u_{i+1}^{n-1/2} - u_{i-1}^{n-1/2} + 2u_i^{n-1/2}}{\Delta x^2 \cdot \Delta t}$$

having considered:

$$\left[\frac{\partial^2 u}{\partial x^2} \right]^{n-1/2} = \frac{u_{i+1}^{n-1/2} + u_{i-1}^{n-1/2} - 2u_i^{n-1/2}}{\Delta x^2}$$

and similarly:

$$\left[\frac{\partial^2 u}{\partial x^2} \right]^{n+1/2} = \frac{u_{i+1}^{n+1/2} + u_{i-1}^{n+1/2} - 2u_i^{n+1/2}}{\Delta x^2}$$

Therefore, we end up with a numerical scheme of the implicit kind. Considering again the initial differential scheme (6) by omitting the non-linear terms and assuming the waves to be of the type $a e^{i(\omega t - k \cdot x)}$ (where a is a certain amplitude), then the solution of the equations shows the dispersion relation to be:

$$\omega^2 = k^2 g \cdot d \left(1 - \frac{1}{3} k^2 d^2 \right), \text{ for } \frac{k^2 d^2}{3} < 1$$

(see Ref 16).

Therefore, the effect of the third order term in the equations is simply to give a correction to the shallow water phase velocity $\sqrt{\omega/k} = \sqrt{g \cdot d}$ and consequently, to the wave length as well. Indeed, due to dispersion relationship above, it will be $C < \sqrt{g \cdot d}$ where C is the new phase velocity and, since, $k = \frac{\omega}{C}$, we will have $k > k_0 (= \frac{\omega}{\sqrt{g \cdot d}})$. Thus, it is evident that the dispersion term in the initial set of differential equations will cause a reduction in the wave length. The numerical solution of the difference scheme (14) described above shows similar behaviour.

For this reason, we choose a channel of constant depth $h_0 = 10.0\text{m}$ along which a train of constant sinusoidal waves is travelling with period $T = 10$ sec and an amplitude of $a = 1.0\text{m}$. The wave length corresponding to shallow water speed $\sqrt{gd} = 9.905\text{m/s}$ is equal to $9.905 \times T = 99.05\text{m}$ and the Ursell parameter $U_r \approx 9$. The implicit numerical scheme derived earlier on is implemented for this particular, simple case under the following conditions (see Fig 3):

- A large number of grid points describes the one-dimensional channel being a distance $\Delta x = 10.0\text{m}$ apart.
- The first grid point is an elevation point (+) where the wave amplitude a is provided as a sinusoidal function of time i.e. $a \sin(\frac{2\pi}{T} \cdot t)$ every $\Delta t = 1.0$ sec. Under the circumstances, the Courant number of the numerical scheme is $C_r = \frac{\sqrt{g \cdot d} \cdot \Delta t}{\Delta x} = 1$ and the numerical error is zero. The third order dispersive term $\frac{\partial^3 u}{\partial x \cdot \partial t^2}$, therefore, is not contaminated by any numerical error of the same order.

The model is run for a substantial number of time steps, so that we get a clear picture of the wave propagation along the channel. Nevertheless, we choose the total length of this channel so that the wave front never reaches its end at the completion of the calculations. In this way, we do not have to apply numerically any end conditions which could introduce errors in the wave profile.

- In the numerical implementation of scheme (14), we are sweeping the solution from left to right, therefore, the continuity equation remains explicit but the momentum equation takes the implicit form:

$$e_i \cdot U_{i-1}^{n+1/2} + f_i \cdot U_i^{n+1/2} + g_i \cdot U_{i+1}^{n+1/2} \equiv 0$$

for each velocity point i along the axis of the channel. The quantities e_i , f_i and g_i depend only on known values of the variables. Considering the total number of points, the above relationship leads to a matrix equation of the form: $\underline{A} \cdot \underline{U} = \underline{B}$ where $\underline{U}$ is the matrix of the velocity values at each grid point and $\underline{A}$ is a tridiagonal matrix. This equation is solved by means of a very simple numerical technique.

The results of running the model for this simple case are presented in Fig 3 where we can easily see that, indeed, the effect of the dispersive term in the solution is the reduction of the wave length. We can measure it to have the expected value of $\sim 92.0m$ while the wave amplitude remains the same as before i.e. $1.0m$.

The same test is repeated to test the performance of a numerical scheme of the predictor-corrector type. This is in fact the sort of FD scheme which is eventually chosen to integrate the full form of the Boussinesq equations in the (x, y, t) domain (Eq 18).

In this case, the numerical solution is implemented in two stages:

- First, the equations are solved by omitting the third order term. The calculated new values of the velocity u are used to make a first approximation of the dispersive term (predictor stage).
- Using the calculated value of the 3rd order, dispersive term $\frac{\partial^3 u}{\partial x^2 \partial t}$ in the equations, we update the values of z and u (corrector stage). This can be repeated for as many times as it is deemed necessary for convergence. For the simple one-dimensional test considered, it was not found necessary to iterate more than twice. The results (Fig 4) are the same as before, i.e. the incident wave is propagated with a smaller phase velocity, and therefore, its wave length is reduced. Theoretically, as the number of internal iterations tends to infinity, the solution should become identical to the one provided by the implicit scheme.

Within the context of linear dispersive equations, a qualitative test of the performance of our numerical scheme can be provided by running a solitary instead of a sinusoidal wave at the model entrance and, then monitoring its propagation along the channel. As it has been mentioned in Section 2.1, this sort of wave is a permanent solution of the KdV equation (and consequently, of the Boussinesq equations for uni-directional flow) succeeding in retaining an undistorted shape during its travelling time by balancing exactly the non-linear with the dispersive effects. If as in the case we examine now, the non-linear terms are excluded, the dispersive effects become absolutely dominant. As a result, different parts of the wave start to travel with different

speeds and, therefore, the solitary wave breaks into a number of waves of different wave length. Such a propagation has, indeed, an analytical solution (see Ref 16) but the purpose of the present experiment does not justify its practical implementation due to the difficulties involved.

Nevertheless, the model results (see Fig 5) bear all the basic characteristics foreseen by analytical considerations:

- At the front, the wave tends to decay exponentially as time goes on.
- At the back, a tail of short waves is formed which are obviously left behind, because they are moving with slower speeds. It is evident, that the farther the wave lies from the front, the shorter its wavelength is. In Fig 5, the instantaneous wave profile is provided by the numerical scheme at different times as the input solitary wave travels and unfolds itself towards the end of the channel. This numerical solution shows clearly all the basic analytical features described above.

3.2 Non-linear conditions

In this case, there is a real dearth of analytical solutions against which we could check the behaviour of the finite difference model. We can use again the propagation of a solitary wave as a basic test, since if the finite difference model simulates accurately both the dispersive and non-linear terms of the equations, then, such a wave should retain its form unchanged as it travels along the axis of the channel. By means of the linear numerical tests of the previous section, we established the ability of the model to integrate correctly the third order dispersive term. It is now required to demonstrate

its ability to integrate correctly the non-linear terms as well and this can be shown by a solitary wave propagating downstream without any shape deformation. We keep the same flow, geometrical and operational details for the test as before, that is:

mean depth = 10.0m, wave amplitude = 1.0m

$\Delta x = 10.0m$, $\Delta t = 1 \text{ sec}$, $C_r = 1$.

The incident solitary wave is provided through the first elevation grid point as a function of time. This function is the solution of the KdV equation as the wave length and period becomes infinite and it is,

$$z = H \operatorname{sech}^2 q$$

where

$$q = \frac{\sqrt{3} \varepsilon}{2d} \left(1 - \frac{5}{8} \varepsilon\right) (x - C \cdot t)$$

$$\varepsilon = \frac{H}{d} \tag{15}$$

$$C = \sqrt{g \cdot d} \left(1 + \frac{1}{2} \varepsilon\right)$$

H = wave amplitude

So far as the dispersive term $\partial^3/\partial x^2 \cdot \partial$ is concerned, we follow the same procedure as before for the linear case i.e. we introduce an implicit FD approximation. The non-linear terms are simulated following the well established, conventional method adopted at HR for explicit schemes describing tidal flows: we approximate the partial derivatives by angled finite differences which leads to the following difference scheme (the dispersion term is temporarily omitted): (in conjunction with Fig 2)

Continuity Equation:

$$z_i^{n+1} = z_i^n - \Delta t \frac{u_{i+1/2}^{n+1/2} \cdot (d+ J_{x_{i+1/2}}(z)) - u_{i-1/2}^{n+1/2} \cdot (d+ J_{x_{i-1/2}}(z))}{\Delta x}$$

where:

$$\begin{aligned} J_{x_{i+1/2}}(z) &= \frac{1}{2} (z_{i+1}^n + z_i^{n+1}) \\ J_{x_{i-1/2}}(z) &= \frac{1}{2} (z_{i-1}^{n+1} + z_i^n) \end{aligned} \quad (16)$$

Momentum equation:

$$u_i^{n+1/2} = u_i^{n-1/2} - g \cdot \Delta t \frac{z_{i+1/2}^n - z_{i-1/2}^n}{\Delta x} - \bar{U}_o \cdot \Delta t \cdot \frac{J_{x_{i+1/2}}(u) - J_{x_{i-1/2}}(u)}{\Delta x}$$

where:

$$\begin{aligned} \bar{U}_o &= \frac{1}{2} (u_{i+1}^{n-1/2} + u_{i-1}^{n+1/2}) \\ J_{x_{i+1/2}}(u) &= \frac{1}{2} (u_i^{n+1/2} + u_{i+1}^{n-1/2}) \\ J_{x_{i-1/2}}(u) &= \frac{1}{2} (u_{i-1}^{n+1} + u_i^{n-1/2}) \end{aligned}$$

By applying the scheme above and calculating the dispersive higher order term as before, we obtain the results presented in Fig 6. The solitary wave does, indeed, retain its shape while propagating along the length of the channel showing that the numerical model simulates successfully all non-linear and dispersive effects.

Similar results are obtained, if we apply a predictor-corrector scheme. In this case, since we

introduce an iterative procedure, there is no need to make recourse to angled approximations to derivatives. That is, we can fully centre our scheme as it is suggested below (in conjunction with Fig 2).

Continuity equation:

$$z_i^{n+1} = z_i^n - \Delta t \frac{u_{i+1/2}^{n+1/2} \cdot \bar{z}_{i+1/2} - u_{i-1/2}^{n+1/2} \cdot \bar{z}_{i-1/2}}{\Delta x}$$

where:

$$\bar{z}_{i+1/2} = \frac{1}{4} (z_i^n + z_i^{n+1} + z_{i+1}^n + z_{i+1}^{n+1})$$

$$\bar{z}_{i-1/2} = \frac{1}{4} (z_i^n + z_i^{n+1} + z_{i-1}^n + z_{i-1}^{n+1})$$

Momentum equation

$$\begin{aligned} u_i^{n+1/2} = u_i^{n-1/2} - g \cdot \Delta t \frac{z_{i+1/2}^n - z_{i-1/2}^n}{\Delta x} \\ - \bar{U}_0 \Delta t \cdot \frac{\bar{U}_{i+1/2}^n - \bar{U}_{i-1/2}^n}{\Delta x} \end{aligned} \quad (17)$$

where:

$$\bar{U}_0 = \frac{1}{2} (u_i^{n+1/2} + u_i^{n-1/2})$$

$$\bar{U}_{i+1/2}^n = \frac{1}{4} (u_{i+1}^{n+1/2} + u_{i+1}^{n-1/2} + u_i^{n+1/2} + u_i^{n-1/2})$$

$$\bar{U}_{i-1/2}^n = \frac{1}{4} (u_i^{n+1/2} + u_i^{n-1/2} + u_{i-1}^{n+1/2} + u_{i-1}^{n-1/2})$$

Again, we can see an input solitary wave to retain its shape as it propagates downstream (Fig 7). Also we find out that the wave travels with the right speed i.e. $C = \sqrt{gd} (1 + \frac{1}{2} \epsilon)$. The question now arises of what would be the solution, if, instead of a solitary wave, we introduced a train of uniform sinusoidal

waves at the entrance. There is experimental evidence (see Ref 7) that when steep, sinusoidal waves with Ursell numbers of a moderate order (i.e. $O(1)$) are generated in a horizontal channel, then the initial wave breaks down into a number of large and small waves. The smaller waves travel more slowly than the larger waves, since the wave speed depends on the amplitude and, therefore, waves periodic in time do not remain simply periodic in space: there is development of "secondary" crests behind the large waves. These large waves are called solitons (waves resembling solitary waves) after an analogous phenomenon in plasma physics. There is, in general, interaction between these solitons and if followed enough, the separated waves will return, periodically in space, to approximate their sinusoidal, initial wave form. These results indicate that this initial sinusoidal wave form produced by the wave generator can be thought of as the forced superposition of a number of solitary waves. Similar behaviour is also exhibited by initially small sinusoidal waves upon entering a shallow shelf.

Madsen, Mei and Salvage (Ref 10) have shown numerically these physical phenomena by solving the appropriate equations and using the method of characteristics. They were able to demonstrate that some of the waves created were resembling, in fact, cnoidal waves than solitary waves and that, in general, the sort of waves produced depends on the value of the Ursell parameter Ur . Their experiments covered the range $2.5 < Ur < 500$ and their conclusions were in brief as follows:

- (a) For small values of Ur , the "secondary" crests take a lot of time to be developed.
- (b) For larger values of Ur , "secondary" crests appear sooner and in greater numbers, forming cnoidal waves at the front (the

faster moving waves of the lot) for moderate U_r values and solitary waves for large (i.e. > 60) values of U_r .

(c) For $U_r > 100$, breaking starts to occur.

Since there is experimental and numerical evidence for such a behaviour as described above, the FD model developed for the solution of the Boussinesq type equations should apparently be able to demonstrate these basic characteristics of the non-linear wave propagation in shallow waters. To this purpose, we examined, for an one-dimensional channel of depth 10.0m, the following two cases of wave features:

- wave amplitude = 1.0m, wave period = 10 secs.
 $U_r \approx 10$
- wave amplitude = 2.5m, wave period = 16 secs.
 $U_r \approx 64$

Running the scheme (17), we produced the results presented in Figures 8 and 9. They are in close agreement with the conclusions of Madsen, Mei and Salvage presented above: in the first case, cnoidal waves are created at the front, while for the second case corresponding to large Ursell values, there are clearly solitary waves moving at the front of the wave train. This is confirmed also by comparing the numerical solution with the solitary wave profile provided by theory (Eq 15) (see Fig 10).

This close agreement of the model results with experimental and theoretical evidence of non-linear wave propagation in shallow waters, shows that the FD scheme behaves in a satisfactory way.

Finally, the same sort of test is repeated for a different kind of input wave: we apply at the

boundary point the two first terms of a Stoke's expansion of a cnoidal wave, i.e:

$$z = a \cos (\omega t - kx) + \frac{3 \omega^2 \cdot a^2}{4g \cdot k^4 d^4} \cos 2(\omega t - kx) + \dots$$

A cnoidal wave is a permanent wave solution of the KdV equation and, thus, also of the Boussinesq type equations (6,13) for uni-directional flow. If we apply at the boundary an infinite number of terms approximating the amplitude z of a cnoidal wave as above, then, the model should be able to retain the permanent wave form of this wave all along the channel in the same way as with a solitary wave. Since we apply only the two first terms of the approximation and not a proper cnoidal wave, we cannot obtain exactly a permanent wave form, because of error waves originating at the entrance of the channel (see Fig 11). Nevertheless, by comparing Fig 11 and Fig 8, the latter presenting the numerical results of applying at the boundary only the first term of the cnoidal wave approximation, i.e. a pure sinusoidal wave, we can note the striking beneficial effect of including the second term towards a permanent wave form propagating downstream. This wave is a solution of the Boussinesq-type equations and it can be reproduced numerical only if the exact solution is initiated at the entrance point of the channel.

All numerical tests so far have been carried out using the version of $\frac{\partial^3 u}{\partial x^2 \partial t}$ of the dispersive term. This version is equivalent to $\frac{\partial^3 z}{\partial t^2 \partial x}$, since they both lead to the same dispersion relation under linear conditions i.e:

$$\omega^2 = \frac{C_0^2 \cdot k^2}{1 + \frac{1}{2} k^2 d^2}$$

which retains always a real ω . The version $\frac{\partial^3 u}{\partial x^2 \partial t}$ is preferred, because it is more tractable to convenient finite differencing.

The other alternative version is $\frac{\partial^3 z}{\partial x^3}$ which may appear more appealing, since it does not involve differentiation with respect to time and, thus, it can lead to straightforward explicit FD schemes. Such a scheme has been applied for linear simulation of a sinusoidal wave travelling along a channel of constant depth 10.0m. The wave characteristics are: wave amplitude equal to 1.0m, period equal to 10 secs. Therefore, the test is identical to the one used for the implicit and predictor-corrector type numerical schemes described in Section 3.1.

The FD approximation [] of the term $\partial^3 z / \partial x^3$ is as follows: (see also Fig 2)

$$\begin{aligned} \left[\frac{\partial^3 z}{\partial x^3} \right]_i &= \frac{1}{\Delta x} \left(\frac{z_{i-1/2} - z_{i+3/2} - 2z_{i+1/2}}{\Delta x^2} \right. \\ &\quad \left. - \frac{z_{i+1/2} + z_{i-3/2} - 2z_{i-1/2}}{\Delta x^2} \right) \\ &= \frac{z_{i+3/2} - z_{i-3/2} - 3z_{i+1/2} + 3z_{i-1/2}}{\Delta x^3} \end{aligned}$$

Running the linear case as before, the solution is the same as with the other schemes, i.e. the effect is the reduction of the wave length (see Fig 12).

Nevertheless, the situation is different if the non-linear terms are added in the equations. The dispersive term $\partial^3 z / \partial x^3$ leads under linear conditions linear conditions to the dispersion relation:

$$\omega^2 = C_0^2 \cdot k^2 \left(1 - \frac{1}{3} d^2 k^2 \right)$$

which can assign both real and imaginary values to the frequency ω . If ω becomes imaginary, then small perturbations in the solution will amplify and thus, this relationship is sensitive to truncation errors and to finite difference approximations. This is the reason why when the non-linear terms are introduced in the difference scheme in the form of finite difference angled approximations (see Section 3.2, Eq (16)), the scheme eventually "blows up" due to the slow growth of initially small errors which tend to amplify in time. Therefore, it is not possible to set up an explicit non-linear scheme with this version of the dispersive term.

On the other hand, since we try to take advantage of the high computing power of the DAP, an explicit scheme with angled approximations to the partial derivatives cannot be implemented in a parallel processing system. All non-linear terms are required to be fully centred and it is for all these reasons that the dispersive term $\frac{\partial^3 u}{\partial x^2 \partial t}$ is eventually adopted in the Boussinesq-type equations. These equations are solved by means of the predictor corrector scheme described in Section 3.2 (Eq 17) and Appendix I.

4 TWO-DIMENSIONAL SOLUTION

In this case, we have to employ the full version of the Boussinesq equations as first described by Peregrine (see Eq 2). All the tests in this report are carried out for channels of constant depth, therefore, these equations take finally the simpler form:

$$\begin{aligned}\frac{\partial z}{\partial t} &= -\frac{\partial}{\partial x} (u \cdot h) - \frac{\partial}{\partial y} (v \cdot h) \\ \frac{Du}{Dt} &= -g \cdot \frac{\partial z}{\partial x} + \frac{1}{3} d^2 \frac{\partial}{\partial x} \frac{\partial}{\partial t} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)\end{aligned}\tag{18}$$

$$\frac{Dv}{Dt} = -g \cdot \frac{\partial z}{\partial y} + \frac{1}{3} d^2 \frac{\partial}{\partial y} \frac{\partial}{\partial t} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)$$

where:

z is the surface elevation

h is total depth i.e. $h = d + z$

$\frac{D}{Dt}$ is total derivative $(= \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y})$

These equations are numerically integrated by means of an FD scheme of the predictor-corrector type - as has been already mentioned - which is described in detail in Appendix I. The problem again arises of how best to validate such a numerical model in view of the complete lack of any analytical solution of the two-dimensional Boussinesq equations.

As a starting point, we think again of the solitary wave and its main characteristic to balance exactly all non-linear terms with the dispersive terms and, thus its ability to retain its initial shape when propagating over a channel of constant depth. If we assume a long-crested solitary wave travelling in the horizontal plane (x,y) , then this wave should be a solution of the two-dimensional Boussinesq-type equations (18).

4.1 Solitary wave

Let us assume a rectangular basin described by the basic DAP unit i.e. by a matrix of 64×64 grid points having all its boundaries open. We consider a long-crested solitary wave entering the basin at 45° through the bottom corner point A (see Fig 13) at the left hand side. The depth of the basin is taken to be 10.0m and the wave height equal to 1.0m. The time increment Δt is chosen to be 1 sec and the spacing of the grid equal to 14m according to the stability criterion (see Appendix I). If the numerical scheme simulates accurately both the non-linear and dispersive terms of equations (18), then it should be

able to retain unchanged the shape of the solitary wave as it propagates along the diagonal AA'. Such a propagation, therefore, could be considered as a means of testing the performance of the FD numerical model. Since the flow is two-dimensional in plan, two open boundary lines are required to initiate the wave motion inside the model area. In reference to Fig 13, we choose these open boundary lines to be line A1 and line A2. Line A1 is in fact a line of v-velocity points, whereas line A2 is a line of u-velocity points. At each of these points, the velocity (v or u respectively) is provided as a function of time by means of the analytical solution, i.e:

$$u = C_0 \cdot \epsilon \cdot \text{sech}^2 q \cdot \cos 45^\circ$$

$$v = C_0 \cdot \epsilon \cdot \text{sech}^2 q \cdot \sin 45^\circ$$

$$\text{where } C_0 = (g \cdot d)^{1/2}, \epsilon = \frac{H}{d}, H \text{ wave height}$$

$$q = \frac{\sqrt{3}\epsilon}{2d} \left(1 - \frac{1}{8}\epsilon\right) (x - C \cdot t) \quad (19)$$

$$C = C_0 \left(1 + \frac{1}{2}\epsilon\right)$$

(see also Section 3.2, Eq 15).

The model is run for a certain number of iterations but we always make sure that the wave does not reach the end limits of the basin area, so that we do not have to apply special boundary conditions. We monitor numerically the propagation of the wave inside the basin and we present the results in the form of surface elevations in a cross-section along the diagonal at various time levels (Fig 13). By comparing the shape of the wave at these levels amongst themselves and also with the theoretical profile, we conclude that this shape remains unchanged during the wave propagation. This provides evidence that the FD model does, indeed, simulate accurately

the non-linear and dispersive terms of the equations and therefore can be considered as a reliable means for the numerical integration of the Boussinesq equations in two spatial dimensions. The same test is repeated for a different angle of incidence i.e. for $\phi = 30^\circ$. The cross-sections at various time levels are now taken not along the diagonal but along a line at 30° to the horizontal x-axis. The results (Figs 14 and 15) verify that the numerical model retains not only the shape of the solitary wave but, also its angle of propagation in the interior of the basin area. We examine now the case where the wave does reach the end limits of the modelled area. In this case, appropriate boundary conditions should be applied there to allow the wave to go through the boundaries without any deformation which could cause reflections at these boundaries and therefore contamination of the proper solution in the interior with numerical errors. We start by considering the simpler case of a solitary wave entering the basin at right angles to the boundary line A2 (see Fig 13). We need to apply the required boundary conditions along the opposite end of the basin i.e. along line A'2 which is chosen to be a line of u-velocity grid points. The velocity u at these points cannot be calculated in the normal way described by the basic FD numerical scheme, since no wave information is available at the outside of the boundary area. Nevertheless, if the Courant number is exactly one, then the characteristic line $\frac{dx}{dt} = \sqrt{g \cdot d}$ originated at point i at time level n. Δt passes exactly through the boundary point i+1 at time level (n+1). Δt under linear conditions. In such a case, the boundary specification is simply that:

$$u_{i+1}^{n+1} = u_i^n \quad (20)$$

Applying such a condition at the end boundary A'2, we succeed in letting the solitary wave through the model area without any deformation whatsoever (see Fig

16). However, for the two-dimensional cases the stability conditions are such that the Courant number $\frac{\sqrt{g.d} \cdot \Delta t}{\Delta x}$ is no longer 1 but $\sqrt{2}/2$. The characteristic line through a boundary point at time $(n+1)$. Δt intersects the time level n . Δt at a point 1 between i and $i+1$. The necessary boundary condition is then:

$$u_{i+1}^{n+1} = u_1^n$$

where $x_1 = -\sqrt{g.d} \cdot \Delta t$ and the value of u_1^n is found by linear interpolation between values u_{i+1}^n and u_i^n . By applying this sort of approximation of the boundary conditions along line A'2, we obtain the results presented in Fig 17. We note again that the solitary wave does pass through the boundary almost totally unhampered and that the numerical error involved is, indeed small. Finally, if we wish the boundary to be partially reflecting, then the boundary condition is formulated as:

$$u_{i+1}^{n+1} = \lambda u_1^n \quad (21)$$

where λ is the reflection coefficient. For $\lambda = 0$, we have total reflection and for $\lambda = 1$, total absorption. Going back to our original test of a solitary wave entering the basin at a 45° angle to the x-axis, we must take into account in the boundary conditions the fact that the wave hits end boundaries at the same angle of 45° . The position of point 1 in the plane (x,y) is defined by the characteristic lines $\frac{dx}{dt} = \sqrt{g.d} \cos 45^\circ$, $\frac{dy}{dt} = \sqrt{g.d} \sin 45^\circ$ and eventually, we find the boundary condition in this case to be:

$$u_{i+1}^{n+1} = \frac{g}{\sqrt{g.d}} \cdot z_{i+1}^{n+1/2} \cdot \cos 45^\circ \quad (22)$$

for a Courant number of $\sqrt{2}/2$.

The solitary wave leaves the model area without any deformation or "secondary" reflections and after a while, the surface inside the basin is completely flat (see Fig 18). The formulation of the boundary conditions above assumes a prior knowledge of the angle of incidence of a wave at an end boundary. This is not in general the case and therefore we have to assess the effect of using in the absorbing boundary conditions an angle which is different from the actual angle of incidence.

We consider solitary waves entering the rectangular basin at 60° to the x-axis, while we retain the form of boundary conditions described above i.e. for an angle of 45° . We present the results as contour lines of surface elevation (Fig 19). There is a certain deformation of the wave at its crest close to the end boundary and in general, the contours do not remain absolutely straight. Numerical errors start to propagate from the absorbing boundary in to the area inside the model. Under the circumstances, it is important that care is exercised in the formulation of the absorbing conditions and the matter should be further investigated. The only way to avoid numerical errors entirely is to know in advance the angle of incidence of the wave at the boundary but this is hardly the case in real situations.

4.2 Reflection test

In a numerical model based on finite differences, any straight coastline is represented in a stepwise manner. The question, therefore, arises whether under these conditions the reflective properties of such boundaries are accurately simulated. To this purpose, we assume a solitary wave travelling from left to right along the x-axis and impinging on a reflector lying at 45° to the x-axis. If the reflection of the wave is total, then, its direction afterwards should

change by 90° . The propagation of the wave is given in Figs 20 and 21 in an isometric plot and we can observe the following:

- (a) Despite its stepwise representation in the model, the straight wall simulates in the proper way the reflection of the solitary wave. Looking at the velocity plot of Fig 21, we can see clearly that the wave is moving parallel to the y-axis after its reflection.
- (b) At the point of reflection itself, the wave height is doubled as expected.
- (c) The portion of the wave which has not been reflected yet continues to propagate undisturbed.

As a further test, the orientation of the wall is altered from 45° to 60° . The direction of propagation after the reflection is again found to be correct (Fig 22).

The obvious conclusion is that a stepwise representation of boundary lines in a FD model has no adverse effects on the wave motion.

Finally, for the same test we introduce a reflection coefficient λ at the wall. In Fig 23, this coefficient is taken equal to $1/2$ whereas in Fig 24, it is taken simply as 1 i.e. the boundary is fully absorbing. For both cases, we apply the boundary conditions which correspond to an angle of incidence of 0° (see Section 4.1, Eq 21).

4.3 Diffraction test

We proceed to test the ability of the finite difference model to produce diffraction effects around

for example a breakwater tip. We choose a harbour area consisting of a narrow entrance L m wide (L is the wave length) which leads to a large expansion in the breakwater lee (see Figs 25 and 26). The southern boundary is taken to be totally reflecting and therefore the diffraction of the waves will be equivalent to the diffraction through a breakwater gap twice the width of the assumed entrance channel, since the solution is symmetrical around this boundary. Consequently, for linear waves we could use for comparison the theoretical solution provided by Sömmersfeld (Ref 15) for this particular case of a breakwater gap.

Unfortunately, there is no theoretical solution for the diffraction of non-linear waves. Nevertheless, if the ratio of wave height to mean water depth is small, then we should expect the non-linear solution to be close to the linear one. We choose the mean depth to be 10.0m and the wave height equal to 1.0m. The time increment is 1 sec and the space increment is found by the stability criterion to be ≈ 14.0 m. Sinusoidal waves of period 10 secs are initiated along the open-boundary line by specifying their u -velocity as a function of time. First, we run the model with the linear version of the equations and without the dispersive terms trying to establish, in principle, the ability of a finite difference scheme to produce diffraction effects around a corner point. The agreement between the numerical results and the theoretical, linear Sömmersfeld solution for a breakwater gap of width $2L$ is shown in Fig 26. The diffraction coefficient contours for the two cases are very close and therefore we can safely assume that the model is able, indeed, to simulate wave diffraction successfully .

At a second stage, the model is run under the same conditions but with its full non-linear and dispersive version. The results are given in Fig 27 and we can

see that the diffraction coefficient curves are slightly different to those corresponding to the linear case. For both the cases, it was found best to apply at the points of exit of the model those boundary conditions which correspond to an angle of incidence of 0° . Finally, we test the diffraction of a solitary wave. For this particular case, there is no analytical solution but some experiments have been carried out in the laboratory which have demonstrated the main features of this physical process (Ref 12). It has been established that once the diffraction of the solitary wave has taken place a trough is created behind it, followed by "secondary" crests. Figure 28 shows that the numerical model produces something similar.

This diffraction process is also combined with a completely closed basin i.e. with a reflecting boundary opposite to the entrance (Fig 28). The results show the formation of a trough behind the wave reflected at the end boundary. This is also found experimentally (Ref 12). The combined diffraction/reflection process in a closed basin presented numerically here shows the same basic features as other numerical solutions of solitary wave movement (see Refs 8 and 13).

5 SET-DOWN WAVE TESTS

All tests so far have considered only one single wave train of a certain frequency. We are interested now to know what happens if waves of different frequencies are propagated within the modelled area. The waves propagate with speeds that depend on their periods and therefore, they will continually move through one another. At certain times a number of them will come together to produce a group of large waves while at other times they will be out of phase giving rise to relatively small waves. When a group of large waves is formed, there is a corresponding increase in the

kinetic energy of orbital water particle movement. This leads to a reduction in the water pressure and if the air pressure is taken to be constant, the result is that a depression in the mean water level occurs between groups of high waves. A compensating rise in the mean level occurs between groups of high waves. This surface perturbation is enhanced by a wave-like flow that develops beneath the surface. This disturbance is known as set-down beneath wave groups and was first described by Longuet-Higgins (Ref 9). It has a periodicity associated with the groups but it differs from a free long wave, because it is tied, in fact, to the wave groups. Therefore it propagates at the group velocity, which is less than the phase velocity of a free long wave of the same period as the set-down. This disturbance with period of the order of minutes is extremely important for large moored vessels, since the natural periods of horizontal oscillation of such vessels on their moorings are typically within the range of 30 sec to 2 minutes. As a result, a significant resonant response of the vessel can be produced by relatively small amplitude long period wave motions, which in certain cases can cause moorings to part. The problem can be compounded for vessels moored inside harbours when long period wave motions are amplified through harbour resonance, since it has been shown both theoretically and experimentally by Bowers (Ref 6) that set-down behaves much like an ordinary long wave when it excites the resonant modes of harbours.

The simplest example of the set-down beneath wave groups is provided by a wave system consisting of just two frequencies. In this case the groups of waves (and consequently the set-down) occur regularly with a frequency equal to the difference between the two wave frequencies.

5.1 One-dimensional tests

We employ the implicit numerical scheme described in Section 3.2, Eq 17. We consider a very long channel of 10m depth and two trains of regular waves travelling along its length with the following characteristics:

Train one: Period $T_1 = 10$ secs, amplitude $a_1 = 0.5$ m

Train two: Period $T_2 = 14$ secs, amplitude $a_2 = 0.3$ m

The inclusion of the dispersive term $\partial^3 u / \partial x^2 \partial t$ in the equations allow the waves to travel with a speed depending on their period and combined with the inclusion of the non-linear terms, it makes possible the formation of the set down wave along the channel. We can predict the main characteristics of this wave by considering the basic differential equations (13); a Stoke's expansion to the second order of the variables involved and subsequent solution of the equations can lead to the following profile for the set-down wave (Ref 5).

$$D \cos (\omega^- t - k^- x) \quad (23)$$

where:

$$\omega^- = \omega_2 - \omega_1$$

$$k^- = k_2 - k_1$$

$$D = \frac{C1}{C2}$$

$$C1 = \frac{k^- \cdot a_1 \cdot a_2}{2d} \left(\frac{k^- \cdot \omega_1 \cdot \omega_2}{k_1 \cdot k_2} + \omega^- \left[\frac{\omega_1}{k_1} + \frac{\omega_2}{k_2} \right] \right)$$

$$C2 = (\omega^-)^2 - g \cdot d \cdot (k^-)^2 + \frac{g \cdot d^3 \cdot k^{-4}}{3}$$

a_1, k_1, ω_1 : amplitude, wave number and angular frequency of wave 1.

a_2, k_2, ω_2 : amplitude, wave number and angular frequency of wave 2.

In general, the second order wave $\eta^{(2)}$ is found to be (Refs 3 and 5):

$$\begin{aligned} z^{(2)} = & \Sigma b_i \cos 2(\omega_i \cdot t - k_i \cdot x) + D \cos (\bar{\omega} t - \bar{k} \cdot x) \\ & + C \cos (\omega^+ \cdot t - k^+ \cdot x) \end{aligned}$$

where:

$$\omega^+ = \omega_2 + \omega_1$$

$$k^+ = k_2 + k_1$$

$$b_i = \frac{3 \omega_i^2 a_i^2}{4g k_i^4 \cdot d^4} \quad i = 1, 2$$

$$C = \frac{3(\omega^+)^2 \cdot a_1 \cdot a_2}{2g k_1 \cdot k_2 (k^+)^2 \cdot d^4}$$

D as before

Under the circumstances, the FD scheme would be able to produce the proper solution of the differential scheme (13), if and only if all the components of the second order wave above are provided at the entrance of the channel i.e. at the first elevation point (+). This is particularly true for the most important of these components i.e. the set-down wave. Experiments already carried out as part of the research programme of HR (Ref 3) have demonstrated that the result of

not programming the wave generator to produce set-down is to introduce long waves with the same period and approximately the same amplitude as set-down but with a phase shift of 180° . It is only in this way that the boundary condition on the paddle face that $u_x^{(2)} = 0$ is satisfied in the absence of secondary paddle movement. As the primary wave system propagates away from the generator, it carries with it the set down associated with wave groups but, also, propagating with the system are secondary long waves. As has been mentioned above, set-down propagates more slowly than the free secondary long waves. They are exactly out of phase at the wave generator and will gradually come into phase with one another with increasing distance from the generator. As the distance increases further the two will again go out of phase and so on. Thus, the response of harbours and moored ships sensitive to long period disturbances could depend on their distance from the wave maker. Exactly the same behaviour is exhibited by a numerical model which does not allow for the set-down correction to be added to the total elevation applied at the first elevation point (+) of the grid (wave maker position) (see Fig 29).

After a certain number of time iterations has elapsed, we carry out spectral analysis based on the Fast Fourier Transform (FFT) method at a number of distances from the channel entrance (Fig 29). We can see that very close to the entrance ($x = 25\text{m}$), the set-down produced by the non-linear and dispersive terms of the model itself is 180° out of phase with the free long wave created due to the conditions at the open boundary and, therefore, they cancel each other. As a result, the spectral analysis reveals that there is no energy at frequency $f_1 - f_2$ at this distance. Moving further down the channel, they separate from each other, since the set-down travels slower with the group velocity instead of the phase

one. This results in a continuous change of the wave energy at frequency f_1-f_2 as we distance from the entrance of the channel. It is noted that the expected value of the set-down amplitude in our case is 0.06m.

In a second run of the model, the set-down wave is added as a function of time to the total wave amplitude imposed at the channel entrance (Fig 30). In the same way as before, we carry out spectral analysis at the same distances from the wave maker. We note, now that the value of wave energy at f_1-f_2 (proportional to the wave amplitude squared) remains almost constant along the length of the channel and does not depend on the distance. The average value of set-down amplitude is found to be 0.061m. In conclusion, we see that for the simulation of the set-down effects it is necessary to add the appropriate second-order corrections to the incident wave.

5.2 Two-dimensional tests

In this case, we make use of data available from laboratory experiments carried out at HR to study the diffraction of set-down waves inside harbours (Ref 4). For this reason, a numerical grid is set-up to cover the entire area of the physical model experiment (Fig 31, Fig 1 of Ref 4). It consists of a narrow, long entrance leading to a large harbour area behind the breakwater arm. Under these conditions, the waves originated at the entrance of the numerical model will travel exactly the same distance as in the physical one, before they reach for the first time the shingle beaches at the end boundaries. These beaches act as energy dissipating mechanisms in the physical model in the sense that they absorb entirely the energy of the shortest waves while they may reflect partially or even totally the energy of longer waves. Numerically, they are represented by a single line of u (or v) grid

points where the appropriate velocity is calculated by applying the proper boundary conditions (see Section 4.1). These conditions ensure that the total energy, both of short and long waves, is "absorbed" through the boundary line. Nevertheless, there are approximations in their finite difference implementation (linear interpolation, no knowledge in advance of the angle of incidence) and therefore, we should expect a certain amount of the wave energy to be reflected back due to numerical errors. It must be noted that all the experiments carried out in this report have shown this amount to be very low and it causes, in fact, no undue distortion of the wave fronts as they exit from the model area.

It is obvious, therefore, that in the numerical model we do not simulate the full action of the shingle beaches of the physical model with their differential absorbing capacity for short and long waves but we simply allow all the waves to go through the boundary limits undistorted and unhampered - at least in theory.

Under the circumstances, it seems that a direct, quantitative comparison between numerical and physical model results is, in general, rather difficult, since waves returning from the boundaries would be different in the two cases and distort in a different manner the diffraction patterns behind the breakwater. Nevertheless, referring to Ref 4 (Table 1), we can see that the experimental conditions of case 3 in the physical model were such as to introduce reflection coefficients well below 10%. It could be used, therefore, as a case study, provided that the numerical model exhibits the same level of wave reflection from the absorbing boundaries. The calculation of the reflection coefficients is an integral part of the whole experiment; not only as a check of the reflection levels inside the model but, principally, because their value is required for the

calculation of the wave amplitude incident on the breakwater tip. This amplitude is in turn needed to define the diffraction coefficients within the harbour area.

The test has been carried out with the following wave characteristics:

	Frequency Hz	Amplitude m	Period sec
Primary wave one	0.07	1.645	14.3
Primary wave two	0.10	1.965	10
Set-down	0.03	0.28	33.33

The motion is initiated inside the model by specifying the total velocity u due to all these three wave components above along the open boundary points (wave paddle position). The space and time increment have been selected to be $\Delta x = 29.0\text{m}$, $\Delta t = 1.18\text{ sec}$ and this means that for the shorter wave we have $N_x \approx 4.8$ points per wave length and $N_t \approx 8.5$ points per wave period. During each run, surface elevation data are stored as a time series at grid points which are identified with wave probe positions in the physical model and, in particular:

- at distances $\frac{3L}{2}$, $\frac{7L}{4}$, $\frac{9L}{4}$, $\frac{5L}{2}$, $\frac{11L}{4}$, $3L$, $\frac{13L}{4}$, $\frac{7L}{2}$ and $3.6L$ from the wave paddle along the breakwater flume (where $L = 513.33\text{m}$ is the wavelength corresponding to the beat frequency $f_1 - f_2$);
- at distances $L/2$, L and $3L/2$ from the breakwater tip along lines positioned at 15° , 30° , 45° , 60° , 75° and 90° from the shadow line (see Fig 31).

Our aim is to define numerically the diffraction pattern of the set-down wave, since the physical experiment has shown it to be different from the diffraction pattern of a free wave having the same frequency and amplitude. The latter has to be established by running the model first with only one wave imposed at its open boundary and having the

characteristics of the set-down wave i.e. $f = 0.03\text{Hz}$, $a = 0.28\text{m}$. After a number of time steps has elapsed (in fact 1006) so that the waves have reached the end boundaries and reflected back in the breakwater flume, we carry out spectral analysis to produce the spectra at the selected wave probe positions (Fig 32). We note, also, that the numerical results presented in Fig 33 suggest that the flow in the flume is more or less one-dimensional. By studying the wave spectra of Fig 32, therefore, we can safely conclude that the wave system in the flume consists of:

- the incident wave
- the reflection of this wave at the absorbing boundaries of the model and
- its re-reflection on the wave paddle at the entrance.

(See Ref 4). As a result, we can proceed in the calculation of the relevant reflection coefficient and phase - as is described in Ref 4 - and finally of the amplitude of the wave incident on the breakwater tip. Using this wave amplitude and the one corresponding to the spectral density calculated by the spectral analysis at each of the points 1 to 7 we can draw the diffraction coefficients curves of Fig 34. They are compared with the theoretical results for linear waves in the case of a breakwater gap and a single arm breakwater. In Fig 34(d), we have a direct comparison between theory, physical and numerical models. Having established by this first run that a reflection of 9% must be expected at the absorbing boundaries for free waves at the frequency and amplitude of the set-down, we proceed in running the model by applying the entire set of incident waves at the open-boundary line i.e. both the two primary waves and their set-down effect.

The wave energy spectra produced in the breakwater flume at frequency $f = 0.03$ Hz are presented in Fig 35. We note immediately that they do not follow the pattern suggested by a one-dimensional flow in the flume as in the first test (Fig 32). There are, also, differences with the physical model results. This happens, because the flow in the flume is now strongly two-dimensional. This is also shown by the results in Fig 36 which presents the free surface elevations in the breakwater flume at iteration number = 800. Under the circumstances, we cannot calculate the reflection coefficients and phase in the straightforward manner as before (Ref 4). Instead, we decide to calculate the diffraction coefficients inside the harbour area by assuming the wave amplitude incident on the tip to be equal to the wave amplitude of the incident wave at frequency $f = 0.03$ at the entrance. This may affect the position but not the shape of the diffraction coefficient curves, and therefore, our general conclusions remain valid.

The diffraction coefficients for each of the primary waves are given in Fig 37 and we can see that their agreement with theoretical values is acceptable.

In Fig 38, the diffraction coefficients of the set-down wave are shown and compared both with theoretical values and, also with the free wave diffraction coefficients obtained previously during the first run of the model (see Fig 34). We note that, despite the fact that the diffraction curves of the set-down and the free wave are not in the same position - for the reasons outlined above - however, their shape is just the same. This suggests that, under the assumed conditions, the diffraction of the set-down does not differ from the diffraction of a free wave having the same amplitude and frequency. One of the reasons for such a behaviour may be that, even before any wave reflecting at the absorbing boundaries has returned back in the flume, the flow

there is already significantly disturbed due solely to the diffraction process at the breakwater tip. This is shown clearly in Fig 39 where the free surface elevations are given along the flume at iteration number = 300. (The wave front needs roughly 200 iterations to reach the absorbing boundaries and 300 to meet the breakwater tip on its way back.) Therefore, the flow in the flume has become strongly two-dimensional with the presence of cross waves (see Fig 40) simply because of the diffraction around the breakwater tip which scatters back waves towards the entrance of the channel.

Finally, in an attempt to remove the cross waves from the numerical solution along the breakwater flume, we enlarged the width of the entrance channel from 140m to approximately 513m i.e. roughly one wave length L . Nevertheless, even in this case the flow remains two-dimensional and the results obtained are the same as before: the set-down diffracts around the breakwater tip in the same way as a free wave having identical characteristics (see Figs 41 and 42).

6 CONCLUSIONS AND RECOMMENDATIONS

A finite difference scheme has been devised to simulate numerically wave propagation in shallow coastal areas. Such wave propagation is mathematically described by the Boussinesq-type equations which introduce the surface elevation z and the mean velocities u and v as the unknown variables. They are basically the well known shallow water equations which are extensively used in tidal and river hydraulics but with extra third order terms in the two momentum equations. These account for vertical accelerations which are still small but not negligible. Under these conditions, the pressure is non-hydrostatic and the waves dispersive i.e. their speed of propagation depends on their period.

The magnitude of these additional terms is comparable with the error usually involved in conventional 2nd order accurate FD techniques for the solution of the shallow water equations. Therefore, particular care must be taken to avoid contamination of these important terms by unacceptable numerical errors. To tackle this significant numerical aspect, we introduce a fully centred FD approximation to the fundamental differential equations and a predictor-corrector approach in calculating all non-linear and dispersive terms.

This was achieved by taking full advantage of the available high power computing capabilities at HR in the form of the DAP system which makes this sort of iterative procedure cost effective and not unduly time consuming.

There are difficulties in the validation of a numerical scheme solving the Boussinesq-type equations, because of a real dearth of simple analytical solutions. We try to make best use of the few available solutions and in the course of the validation process the following conclusions are drawn:

1. The FD numerical model of the predictor-corrector type is found to be able to simulate accurately all non-linear and higher order terms of the Boussinesq equations which are responsible for the introduction of the dispersion effects in the wave propagation.
2. The stability criterion is the same as the one required in tidal flow simulations i.e:

$$\Delta t \leq \Delta x / \sqrt{2gd + \text{max velocity}}).$$

Nevertheless, there is a limit in the range of application of the Boussinesq equations which in practical terms is reflected in the additional condition $\Delta x > 1.15 d$. Courant numbers should be close to unity for accuracy reasons.

3. The model is found capable of reproducing successfully diffraction effects around breakwaters for both linear and non-linear waves; also, it simulates successfully any reflection properties (total or partial) of boundary lines which are represented in the model in a stepwise manner.
4. The one-dimensional boundary conditions applied along artificial boundaries with the view to "absorbing" the wave energy by allowing the waves to leave unhampered through the model boundaries, have caused no undue distortion of the numerical solution for all the examined cases. However, this matter should be further investigated by varying the angle of alignment of a boundary to the direction of the waves.
5. Following similar methodology to physical model experiments carried out at HR in the past, a proper representation of the set-down wave along a flume has been achieved by introducing the necessary second order corrections to the wave profile at the open boundary points. However, the diffraction of this wave around a breakwater tip was found to be similar to the diffraction of a free wave with the same characteristics in all the examined cases. This contrasts with the physical model results and it should be further investigated.

6. Finally, the model could be safely used for the calculation of non-linear wave propagation inside harbours of very simple geometry and constant depth.

The present work is seen as a first important step towards the full development of a finite difference model solving the Boussinesq-type equations by taking into account more realistic input conditions and in particular:

- (a) small variations in the mean water depth and, thus, wave refraction effects;
- (b) more sophisticated boundary conditions along "absorbing" boundaries for concurrent wave trains of different direction;
- (c) random and multi-directional incident waves at harbour entrances.

7 ACKNOWLEDGEMENTS

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APPENDIX I

The various terms in Eq 18 are approximated by centred finite differences in the following manner (in conjunction with Fig 2):

Continuity equation

([] stands for "FD approximation")

$$\left[\frac{\partial z}{\partial t} \right] = \frac{z_{i,j}^n - z_{i,j}^{n-1}}{\Delta t}$$

$$\left[\frac{\partial}{\partial x} (u \cdot d) \right] = \frac{u_{i+1/2,j}^{n-1/2} \cdot \bar{z}_{i+1/2,j} - u_{i-1/2,j}^{n-1/2} \cdot \bar{z}_{i-1/2,j}}{\Delta t}$$

where:

$$\bar{z}_{i+1/2,j} = d + \frac{1}{4} (z_{i,j}^{n-1} + z_{i,j}^n + z_{i+1,j}^n + z_{i+1,j}^{n-1})$$

and similarly for $\bar{z}_{i-1/2,j}$.

$$\left[\frac{\partial}{\partial y} (v \cdot d) \right] = \frac{v_{i,j+1/2}^{n-1/2} \cdot \bar{z}_{i,j+1/2} - v_{i,j-1/2}^n \cdot \bar{z}_{i,j-1/2}}{\Delta y}$$

where:

$$\bar{z}_{i,j+1/2} = d + \frac{1}{4} (z_{i,j}^n + z_{i,j}^{n-1} + z_{i,j+1}^n + z_{i,j+1}^{n-1})$$

and similarly for $\bar{z}_{i,j-1/2}$.

u(v) momentum equation

$$\left[\frac{\partial u}{\partial t} \right] = \frac{u_{i,j}^{n+1/2} - u_{i,j}^{n-1/2}}{\Delta t}$$

$$\left[u \frac{\partial u}{\partial x} \right] = \bar{u} \cdot \frac{u_{i+1,j+1/2}^{n+1/2} + u_{i+1,j-1/2}^{n-1/2} - u_{i-1,j+1/2}^{n+1/2} - u_{i-1,j-1/2}^{n-1/2}}{4 \Delta x}$$

where:

$$\bar{u} = \frac{1}{2} (u_{i,j}^{n+1/2} + u_{i,j}^{n-1/2})$$

$$\left[v \frac{\partial u}{\partial y} \right] = \bar{v} \cdot \frac{u_{i,j+1}^{n+1/2} - u_{i,j-1}^{n+1/2} + u_{i,j+1}^{n-1/2} - u_{i,j-1}^{n-1/2}}{4 \Delta y}$$

where:

$$\bar{v} = \frac{1}{8} (v_{i,j}^{n+1/2} + v_{i,j+1}^{n+1/2} + v_{i+1,j+1}^{n+1/2} + v_{i+1,j}^{n+1/2} + v_{i,j}^{n-1/2} + v_{i,j+1}^{n-1/2} + v_{i+1,j+1}^{n-1/2} + v_{i+1,j}^{n-1/2})$$

$$\left[\frac{\partial z}{\partial x} \right] = \frac{z_{i+1/2,j}^n - z_{i-1/2,j}^n}{\Delta x}$$

$$\left[\frac{\partial^3 u}{\partial x^2 \partial t} \right] = \frac{1}{\Delta t} \left(\frac{u_{i-1,j}^{n+1/2} + u_{i+1,j}^{n+1/2} - 2u_{i,j}^{n+1/2}}{\Delta x^2} - \frac{u_{i-1,j}^{n-1/2} + u_{i+1,j}^{n-1/2} - 2u_{i,j}^{n-1/2}}{\Delta x^2} \right)$$

$$\left[\frac{\partial^3 v}{\partial x \partial y \partial t} \right] = \frac{1}{\Delta t} \left(\frac{v_{i+1/2,j+1/2}^{n+1/2} - v_{i+1/2,j-1/2}^{n+1/2} - v_{i-1/2,j+1/2}^{n+1/2} + v_{i-1/2,j-1/2}^{n+1/2}}{\Delta x \cdot \Delta y} - \frac{v_{i+1/2,j+1/2}^{n-1/2} - v_{i+1/2,j-1/2}^{n-1/2} - v_{i-1/2,j+1/2}^{n-1/2} + v_{i-1/2,j-1/2}^{n-1/2}}{\Delta x \cdot \Delta y} \right)$$

In an analogous manner, we formulate the FD approximately of the v-momentum equation.

Under the conditions, the FD scheme has an implicit form and stability analysis shows that the CFL criterion

$$\Delta t \leq \frac{\Delta x}{(\sqrt{2}g_d + \text{max velocity})}$$

needs to be satisfied.

Stability analysis of the one-dimensional scheme with the term $\partial^3 h / \partial x^3$ in the momentum equation shows that the additional condition

$$\Delta x > \sqrt{\frac{4}{3}} d \approx 1.15 d$$

must be satisfied, so that the dispersion relation introduces always real values of the angular frequency ω . Experimenting with the two-dimensional FD scheme of the predictor-corrector type, we find out by trial and error that the same condition needs to be satisfied together with the CFL criterion in order to ensure stable numerical solutions.

FIGURES

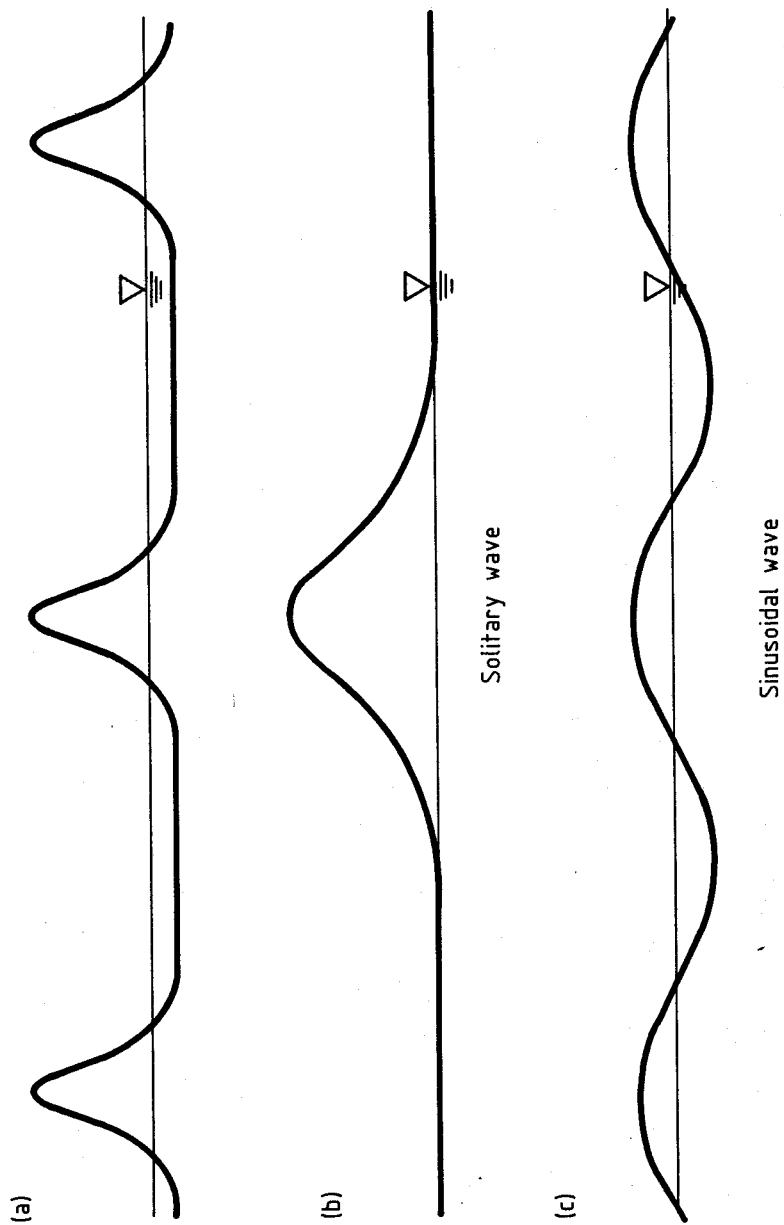


Fig 1 The cnoidal wave profile and its limiting forms

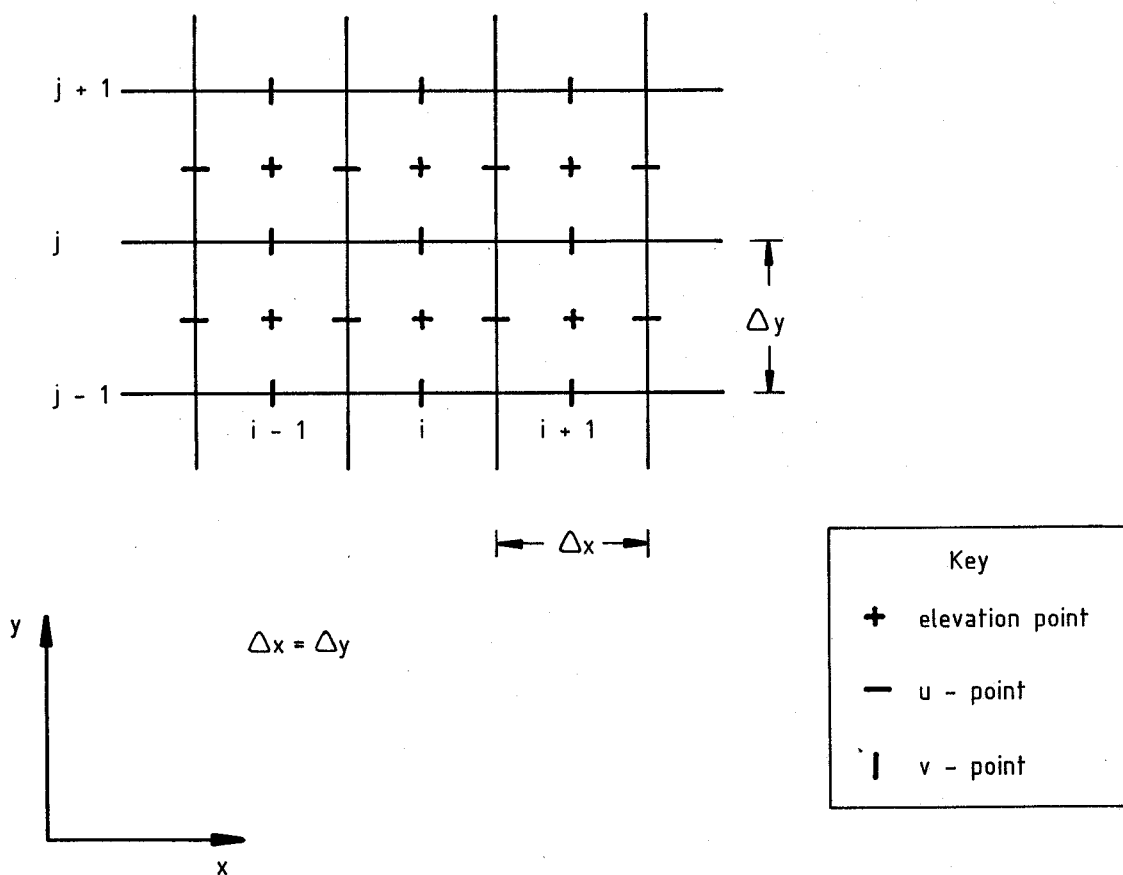


Fig 2(a) Grid layout in space

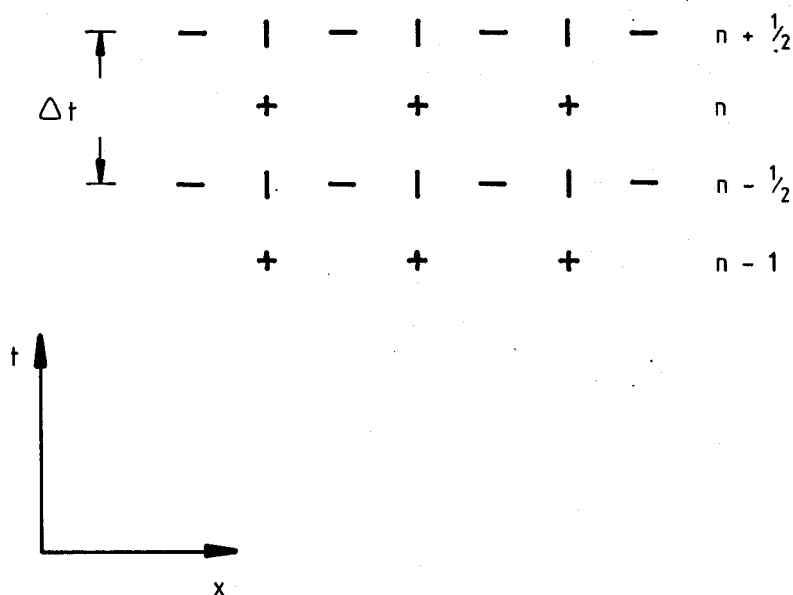


Fig 2(b) Grid layout in time

Fig 2 Basic finite difference grid

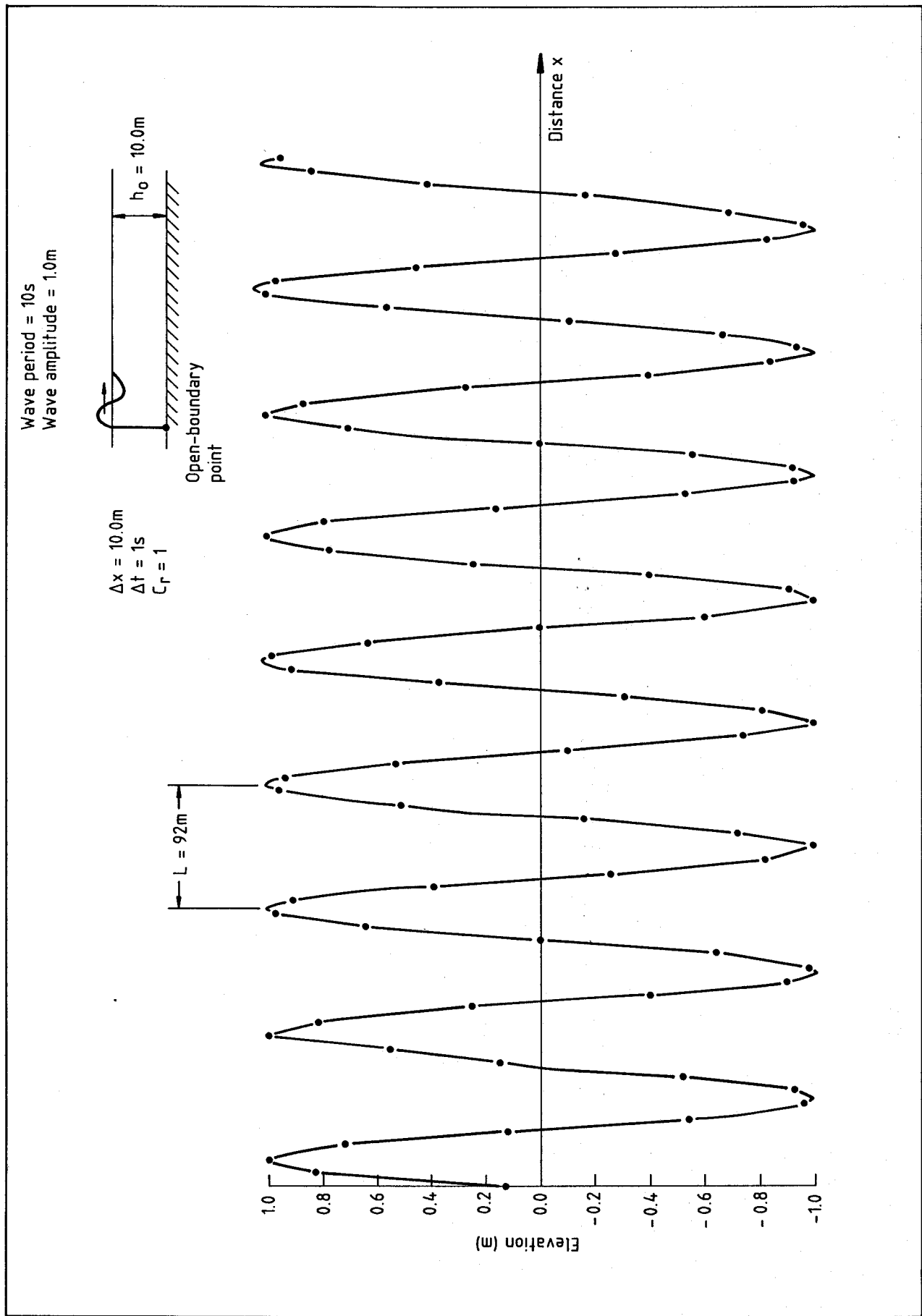


Fig 3 Reduction of wavelength due to dispersion - Linear implicit scheme

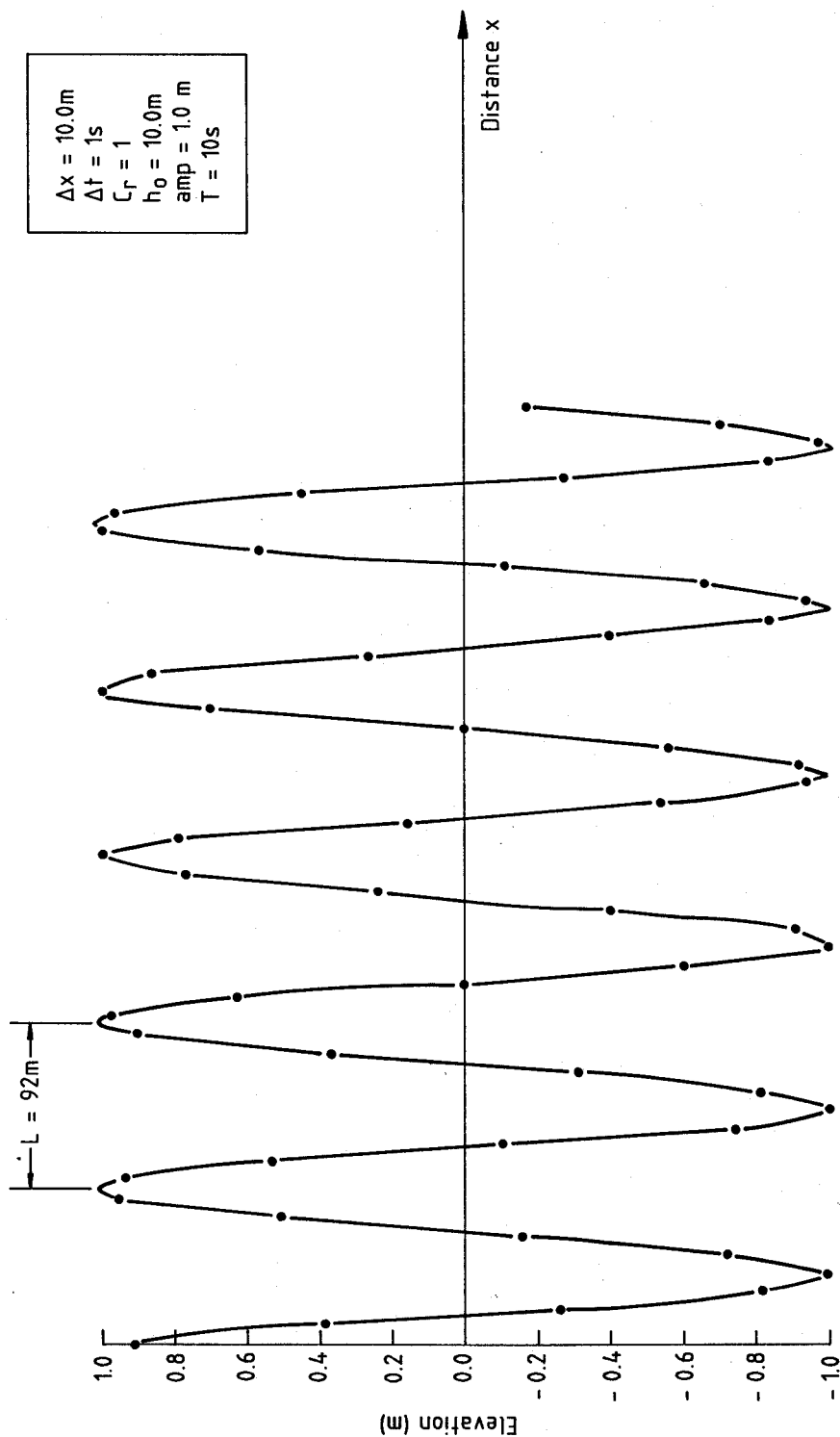


Fig 4 Reduction of wavelength due to dispersion - Predictor/corrector scheme

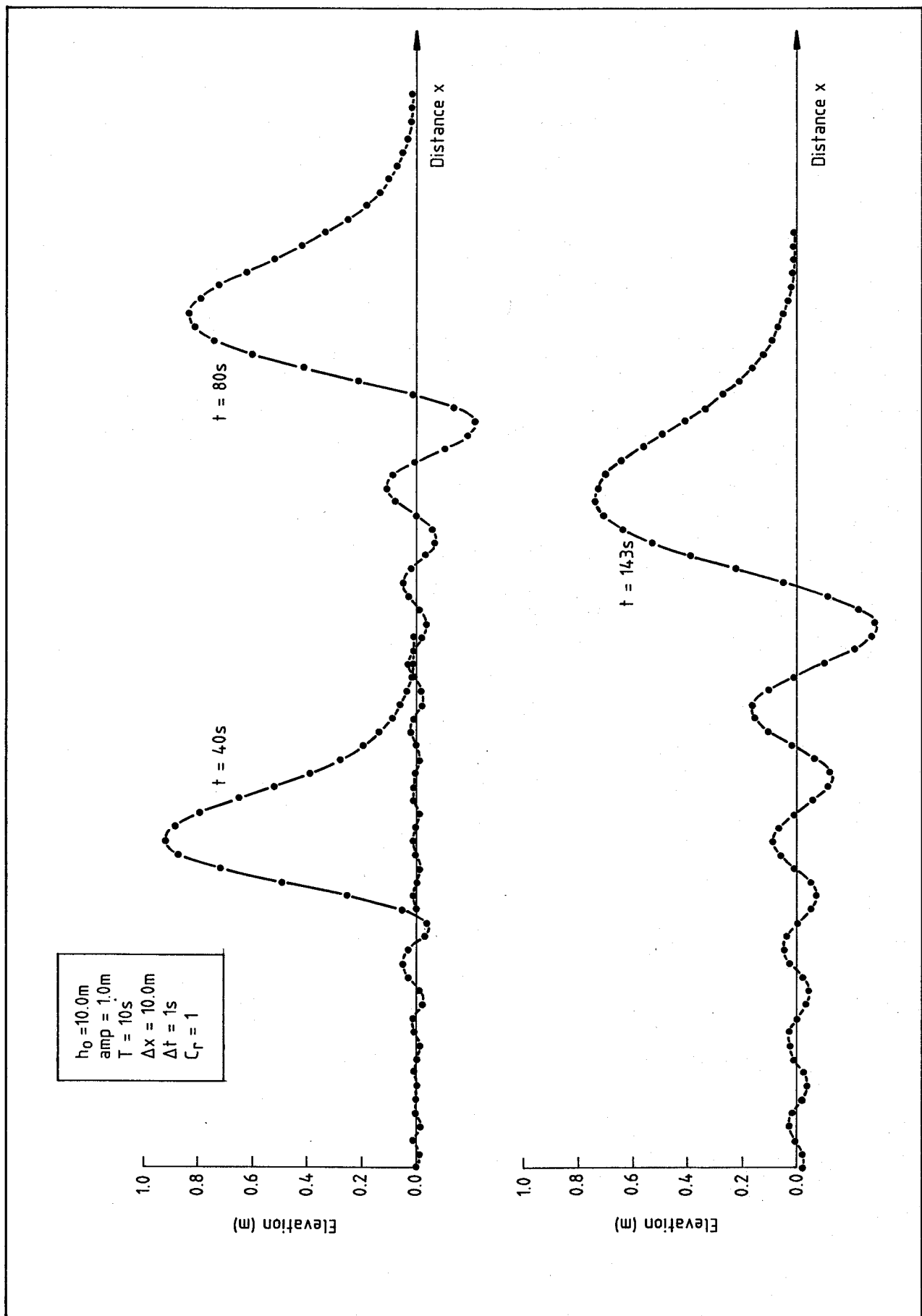


Fig 5 Dispersive evolution of a solitary wave

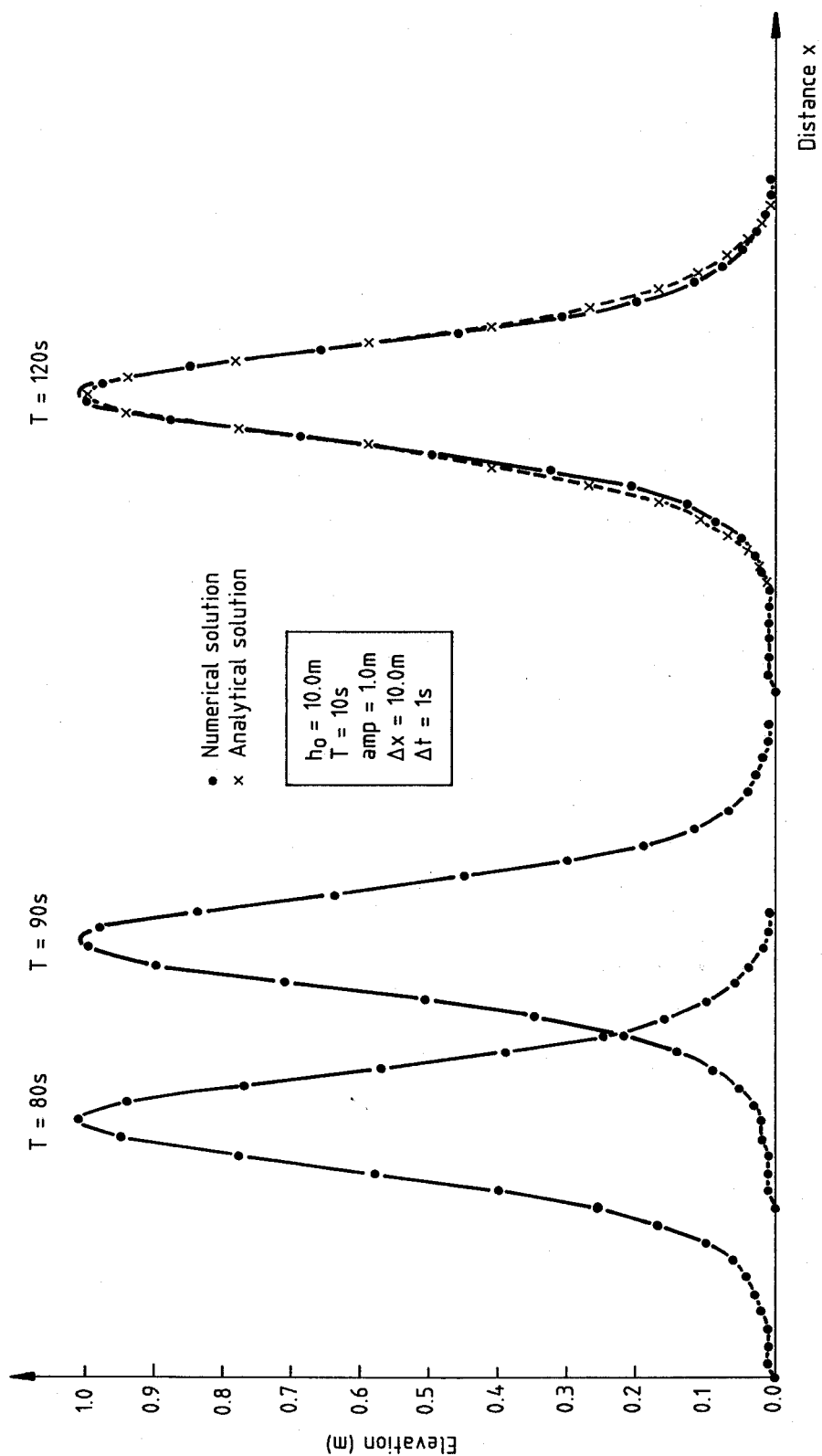


Fig 6 Travelling solitary wave - Implicit scheme

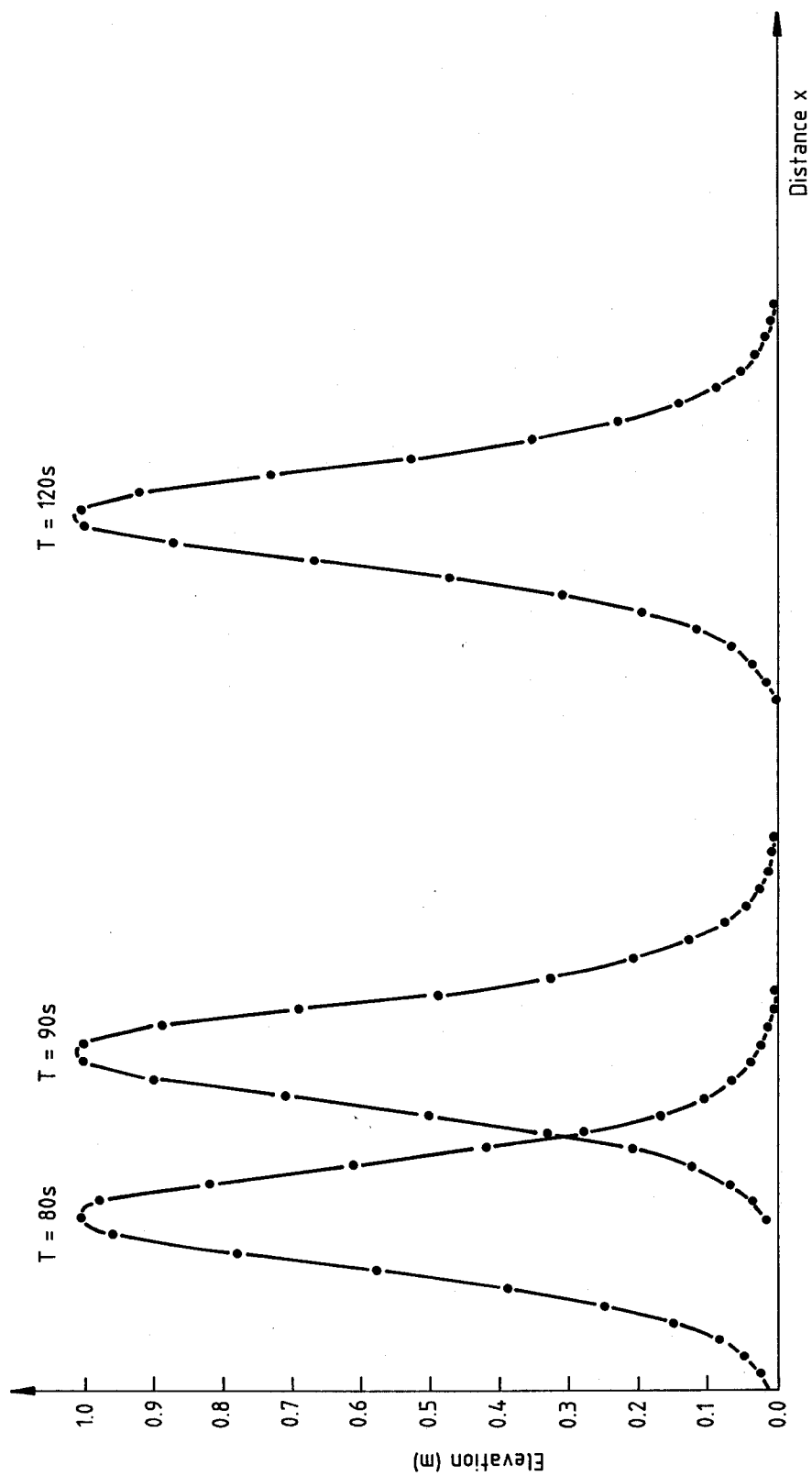


Fig 7 Travelling solitary wave - Predictor/corrector scheme

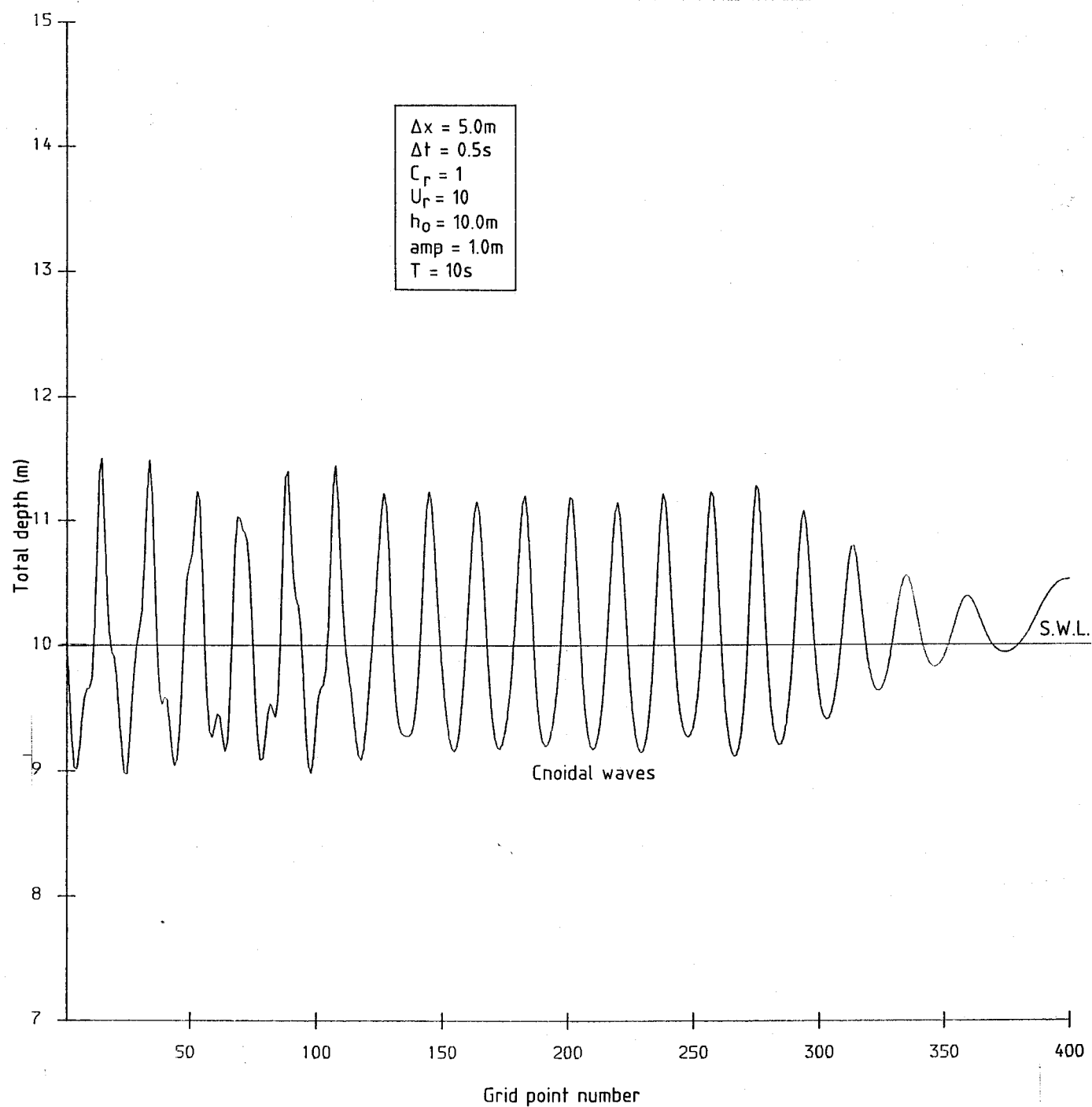


Fig 8 Production of cnoidal waves by a sin wave input

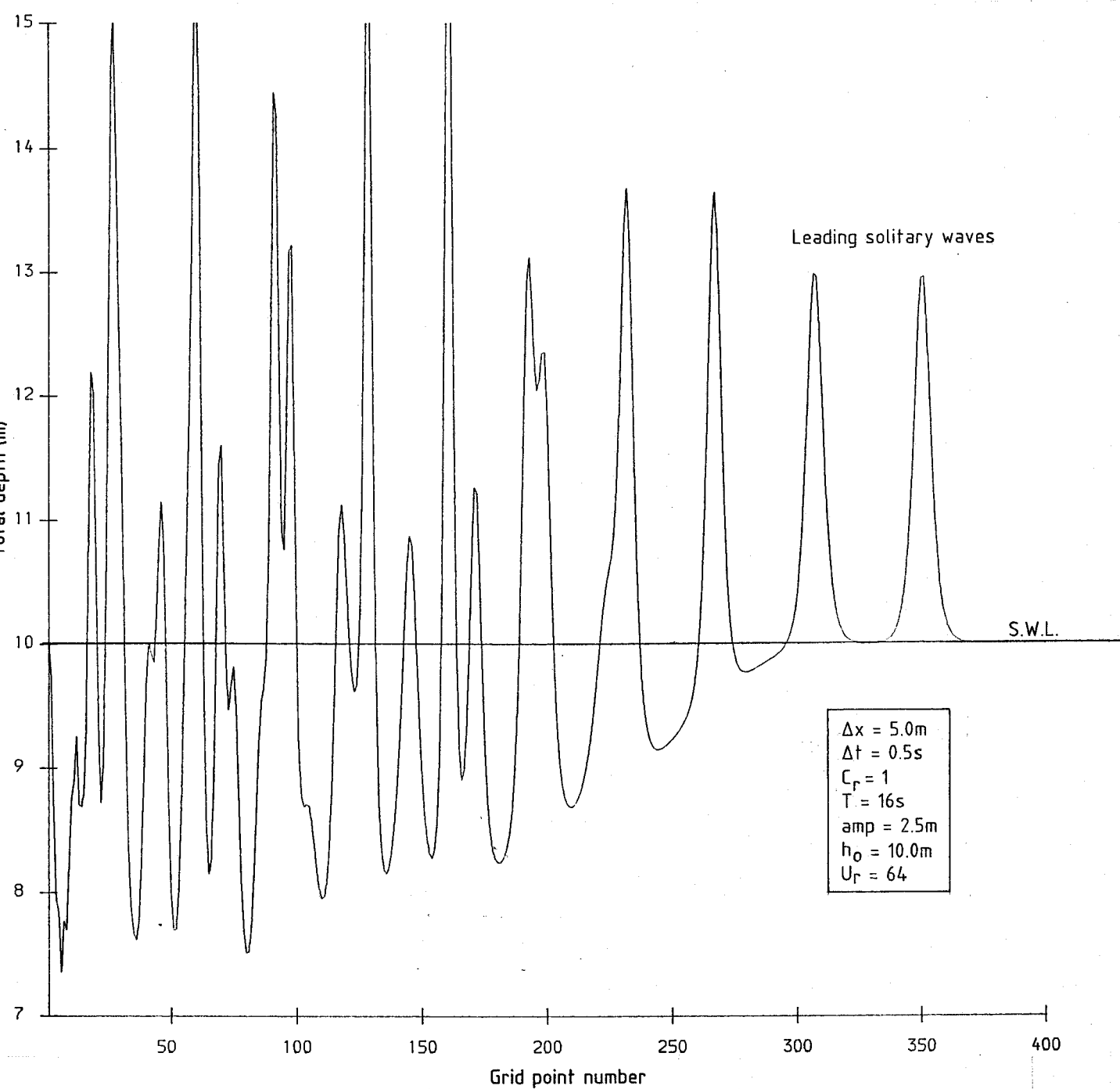


Fig. 9 Production of solitary waves by a sin wave input

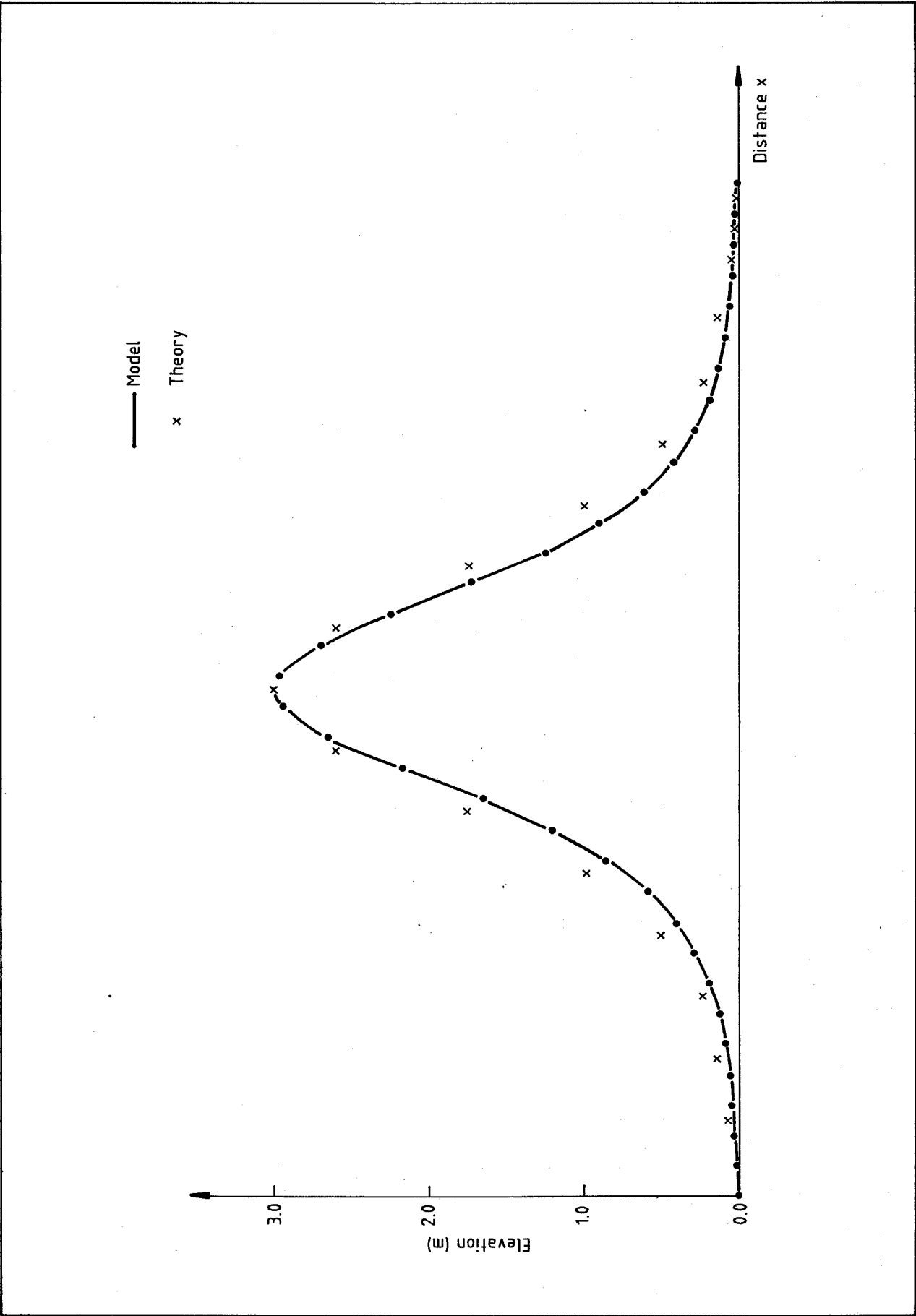


Fig 10 Comparison between model and theory - Solitary wave

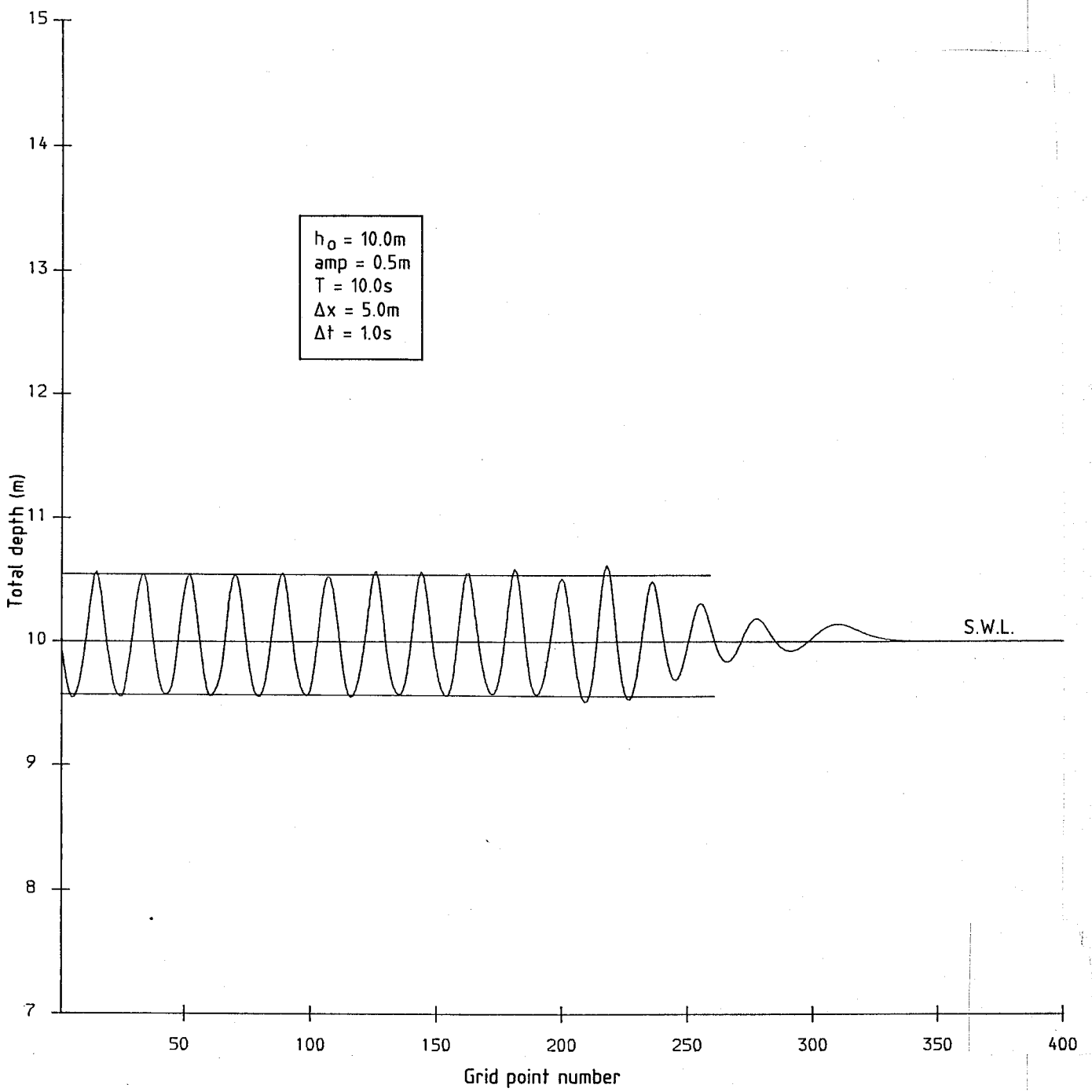


Fig 11 Permanent Cnoidal wave profile.

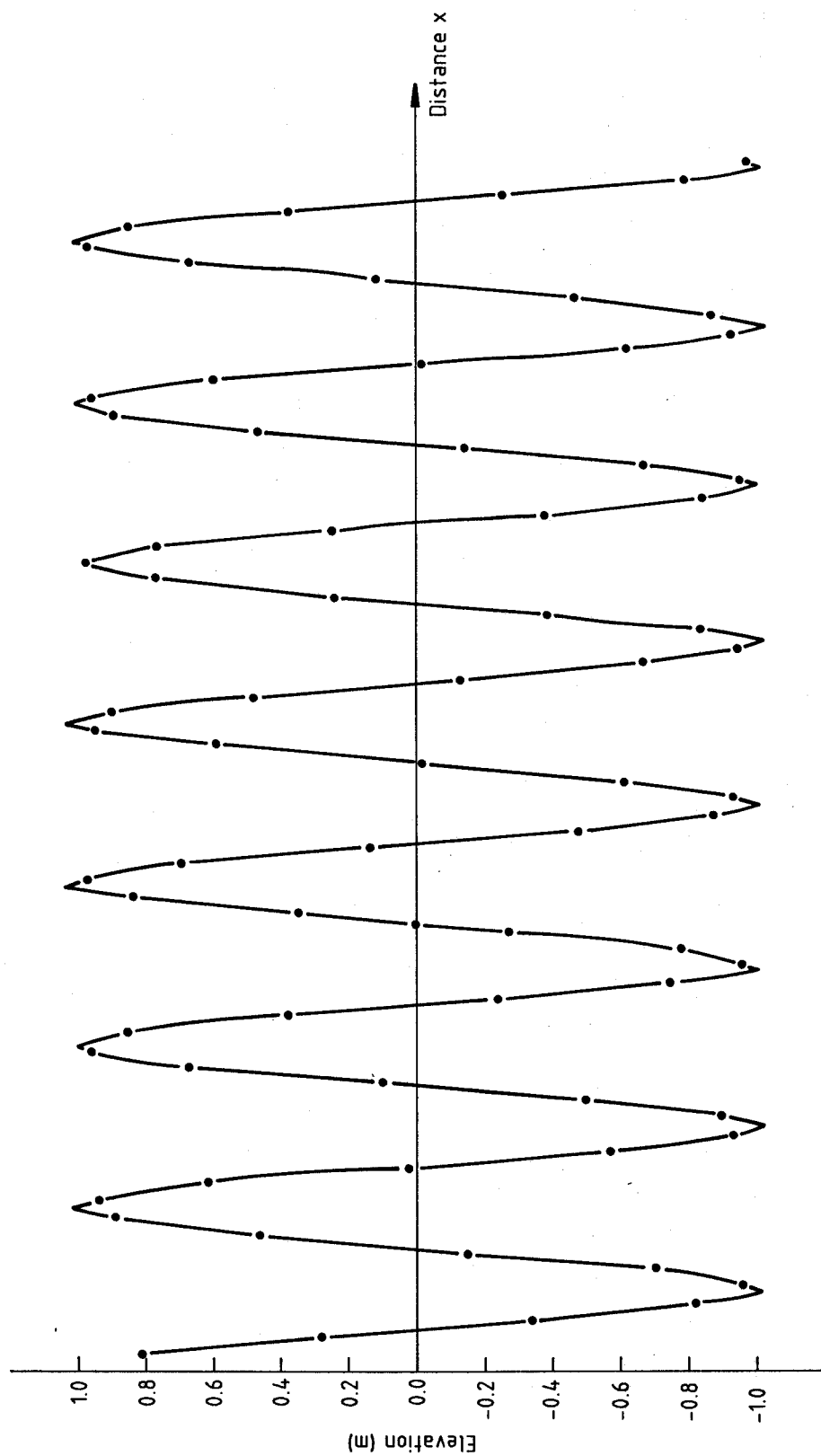


Fig 12 Reduction of the wavelength due to the dispersion term z_{xxx}

$\Delta x = 14.0\text{m}$
 $\Delta t = 1\text{s}$
 $h_0 = 10.0\text{m}$
 $\text{amp} = 1.0\text{m}$

Scale $\overline{\quad 40\text{m} \quad}$
 o Theoretical solution

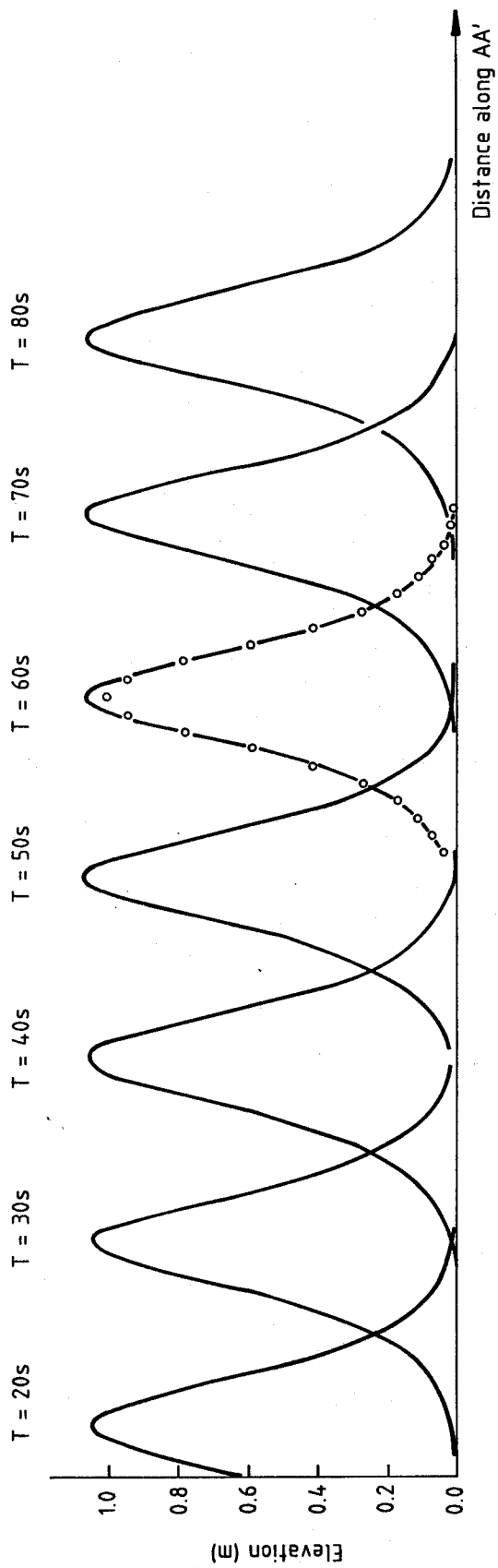
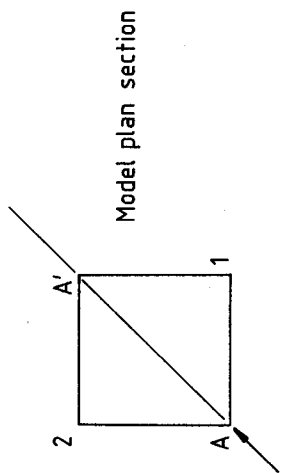
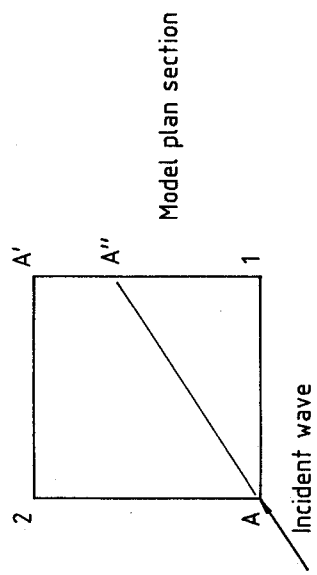


Fig 13 Propagation of a solitary wave - 2D predictor/corrector scheme



$h_0 = 10.0\text{m}$
 $\text{amp} = 1.0\text{m}$
 $\phi = 30^\circ$
 $\Delta x = 14.0\text{m}$
 $\Delta t = 1.0\text{s}$

Scale $\overline{40\text{m}}$
 o Theoretical solution

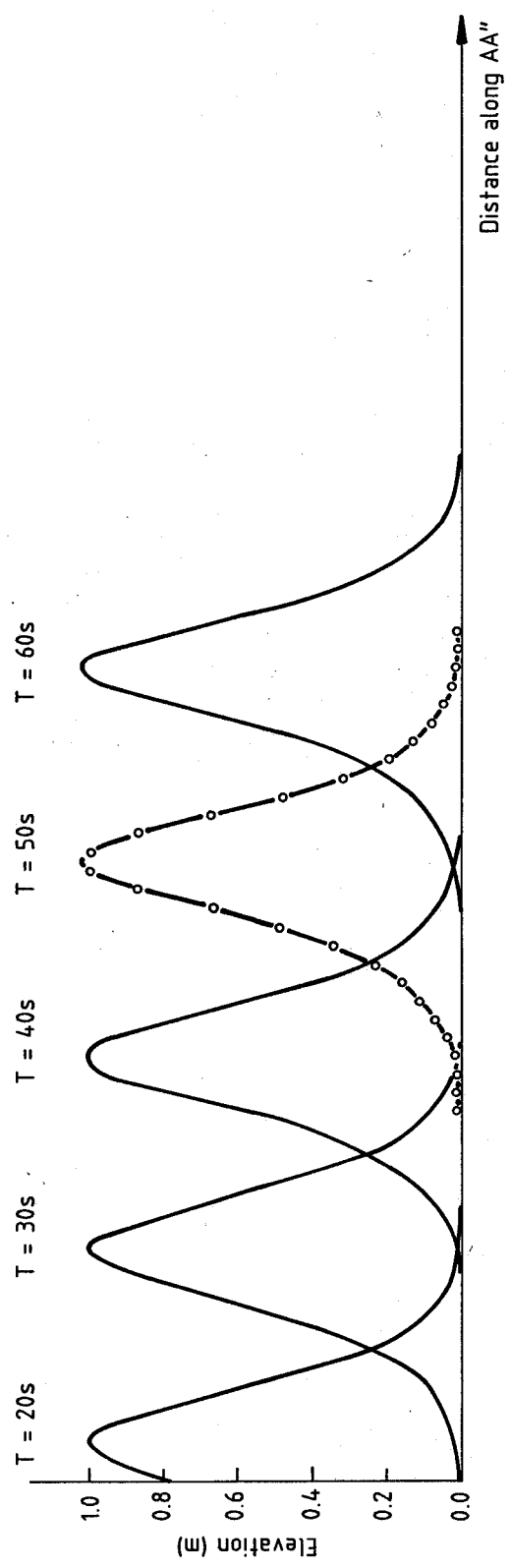


Fig 14 Propagation of a solitary wave at 30°

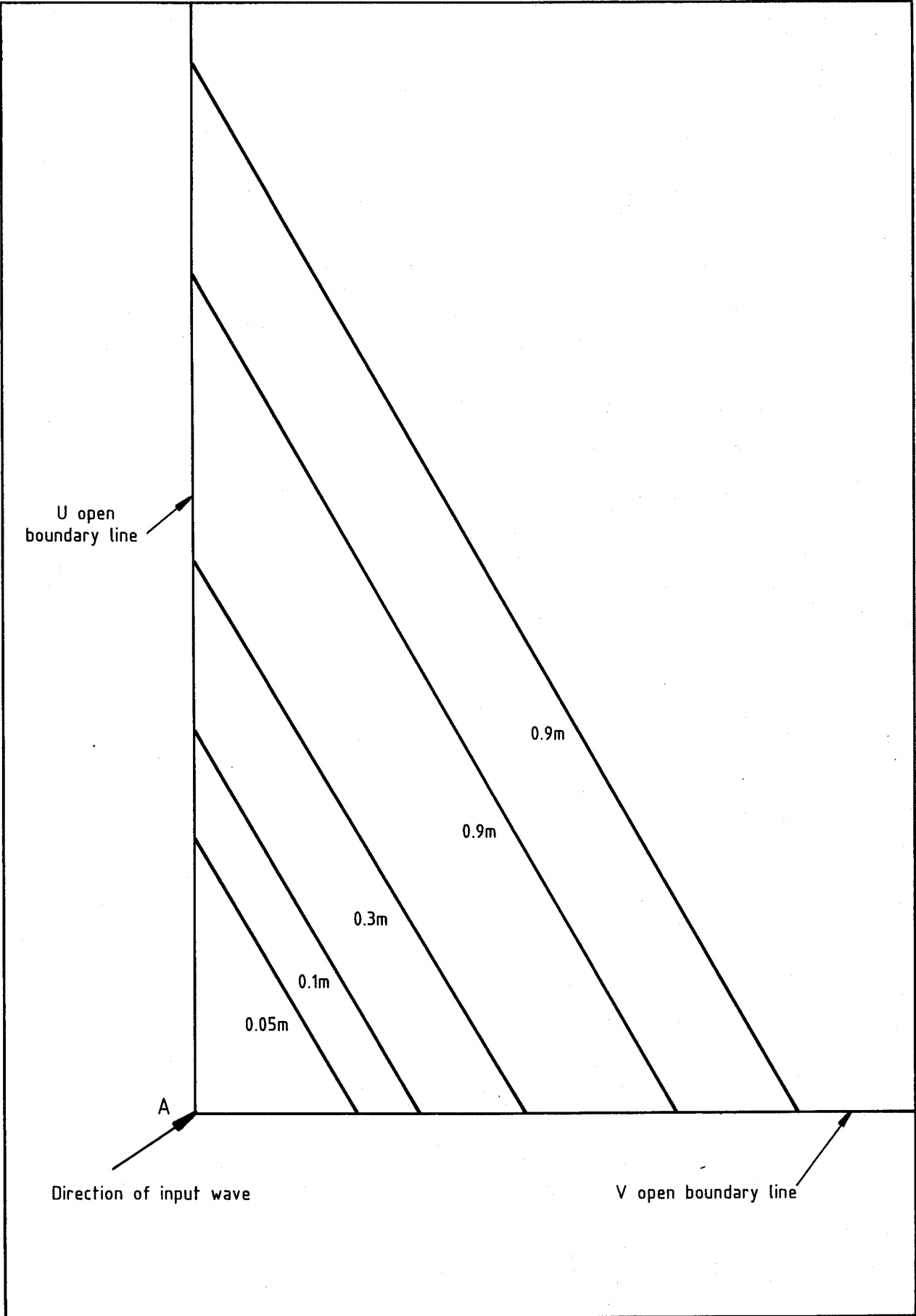


Fig 15 Surface elevation contours -
30° propagation of a solitary wave

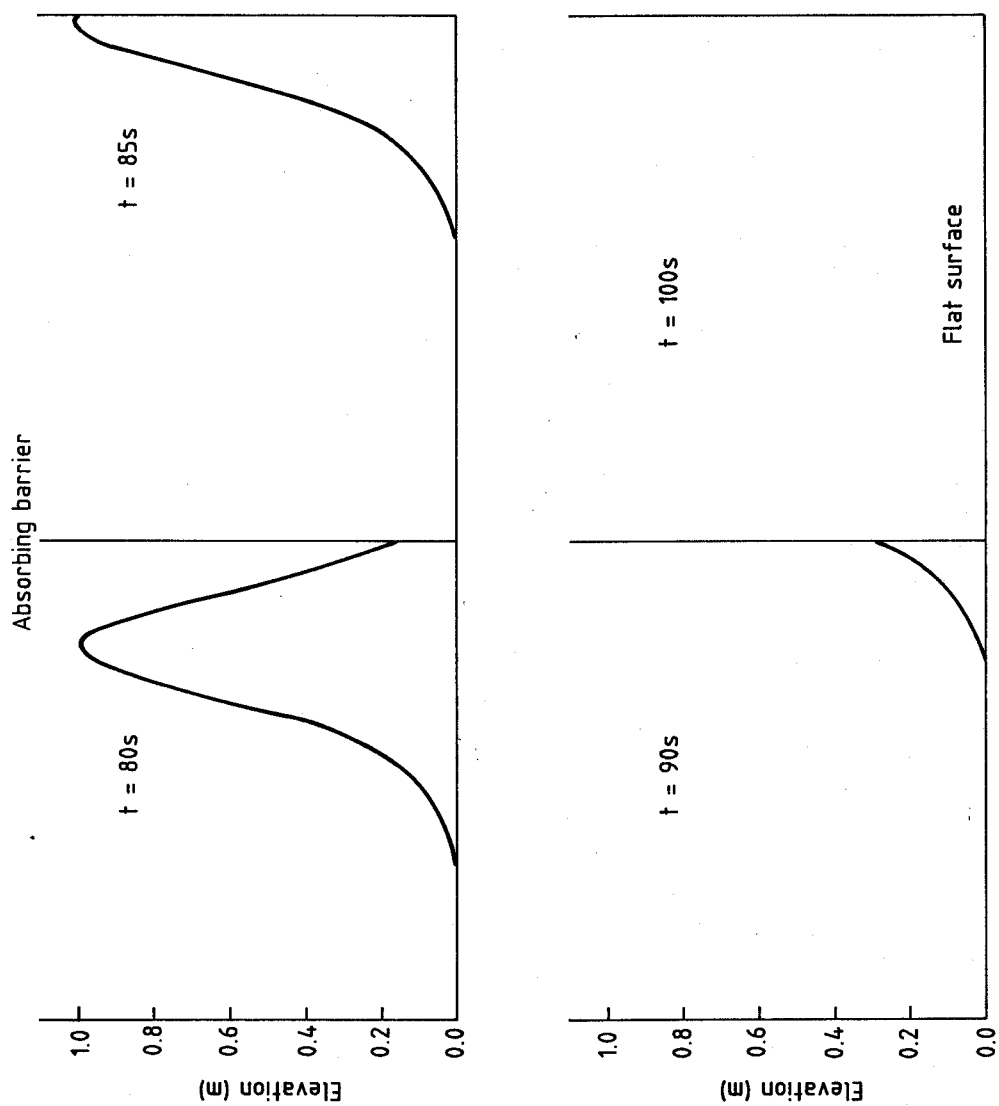


Fig 16 Absorption of a solitary wave - $C_r = 1.0$

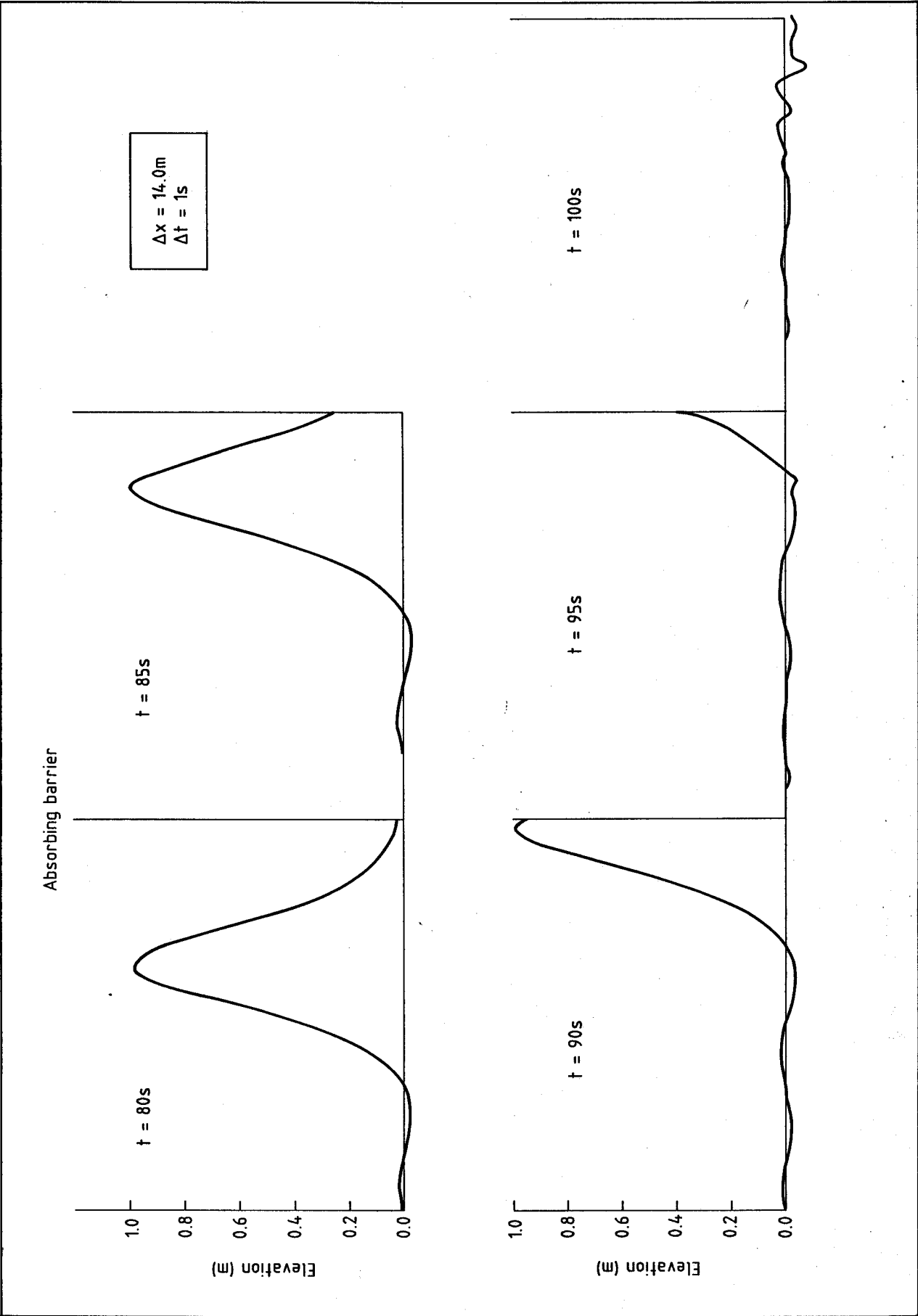


Fig 17 Absorption of a solitary wave - $C_f = 0.707$

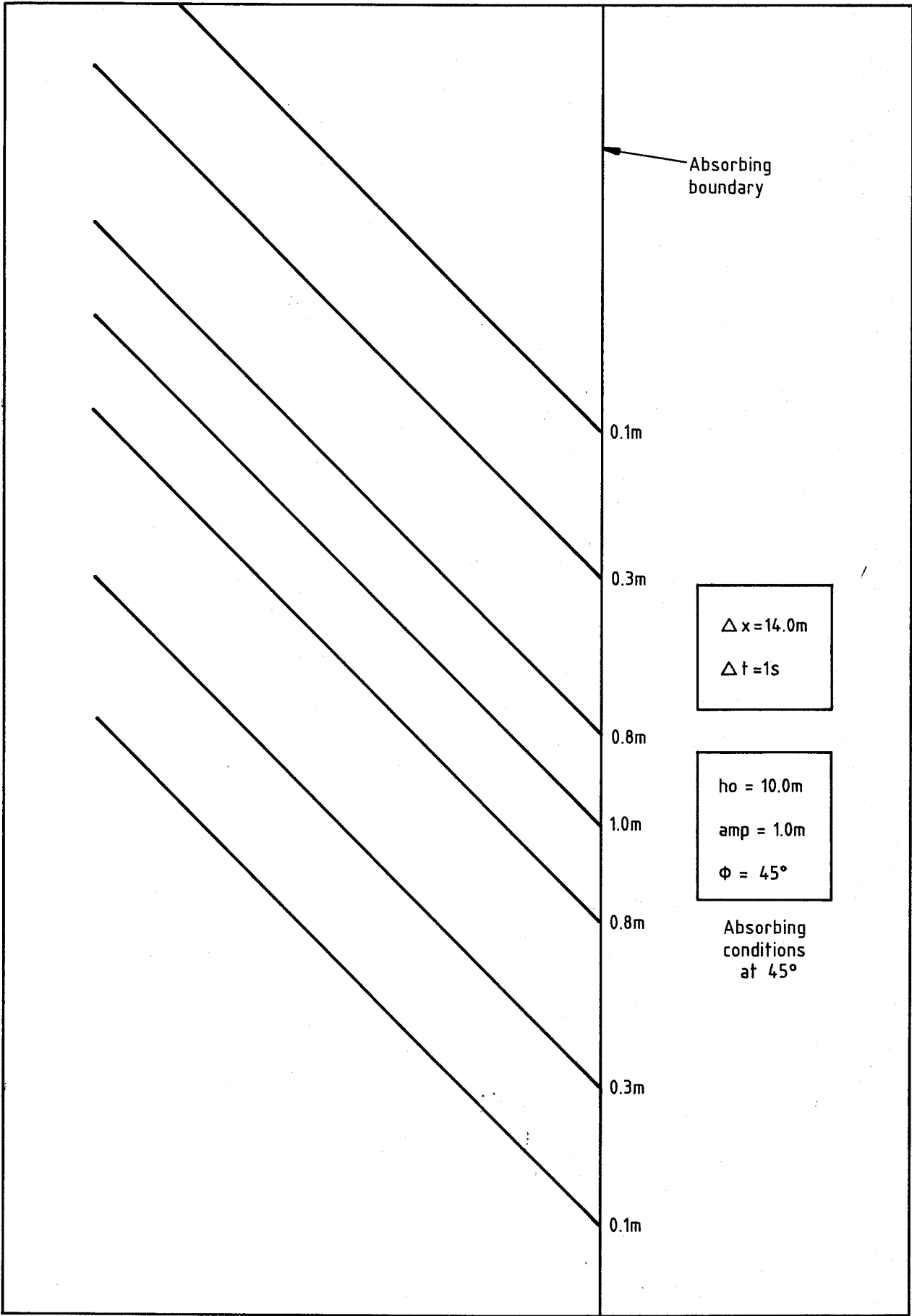


Fig 18 Surface elevation contours - propagation of a solitary wave at 45°

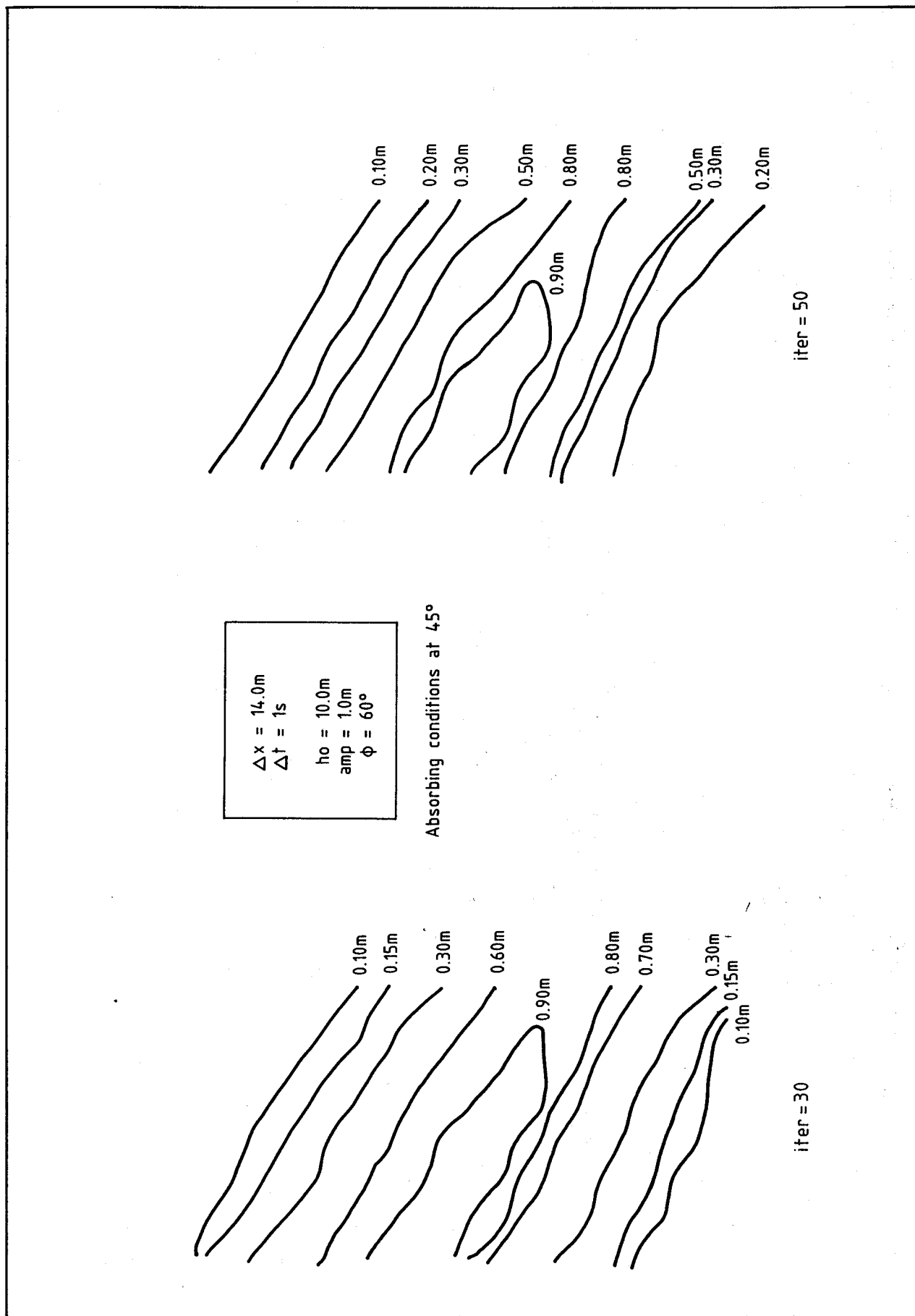


Fig 19 Surface elevation contours - propagation of a solitary wave at 60°

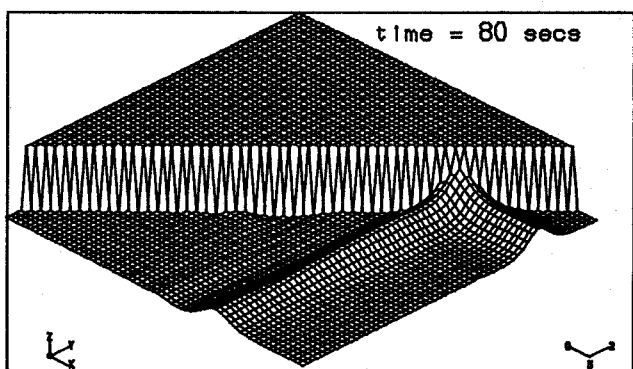
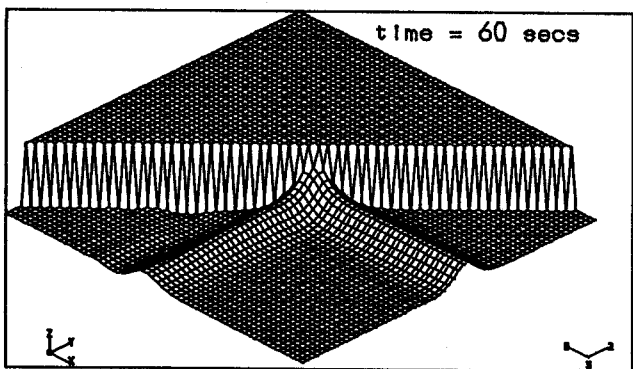
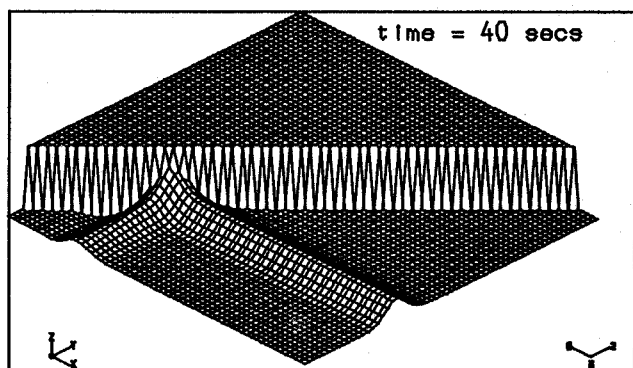
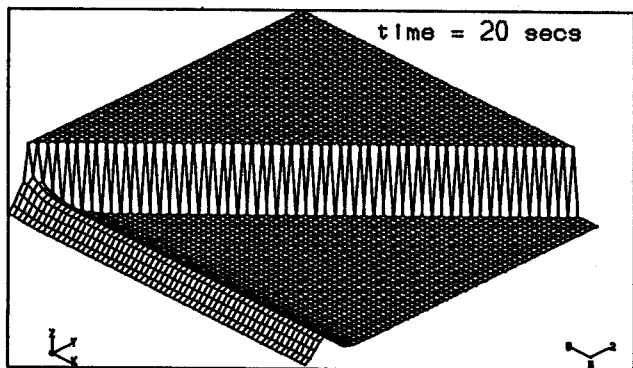


FIG 20 Total reflection of a solitary wave
 $d=10\text{m}$ $a=1\text{m}$ $DX=14\text{m}$ $DT=1\text{s}$ Wall at 45 deg

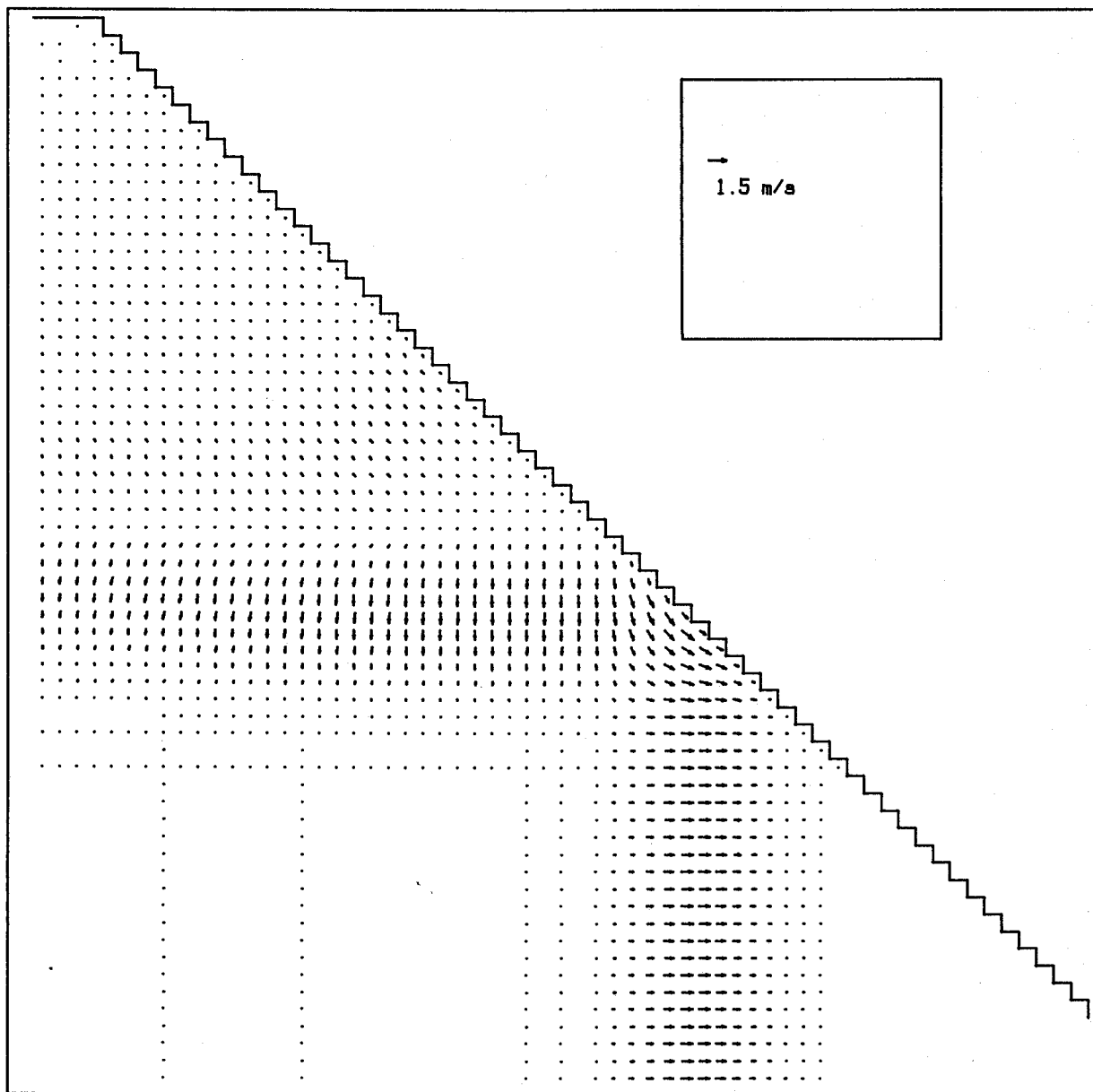


FIG 21 NON-LINEAR SOLITARY WAVE
REFLECTOR AT 45

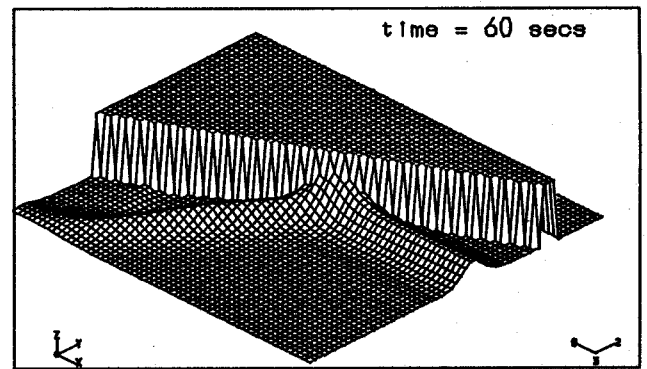
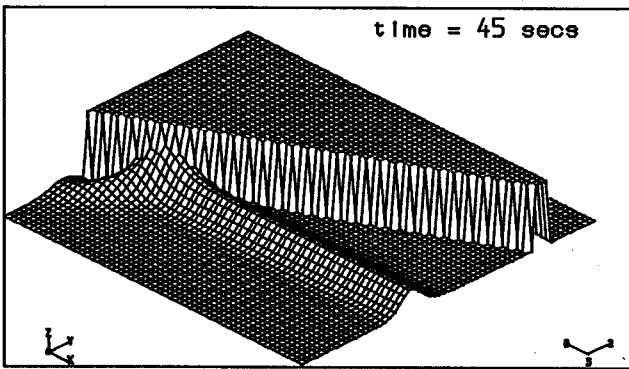
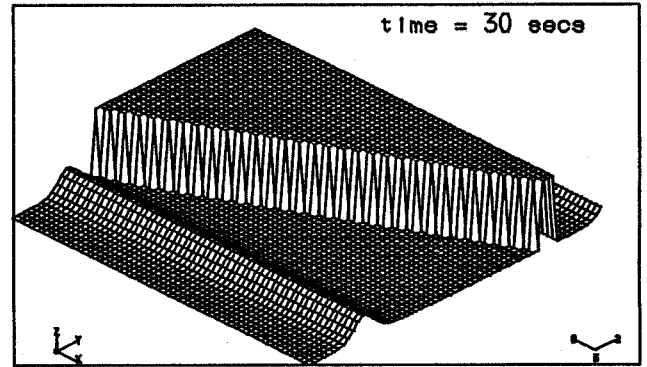
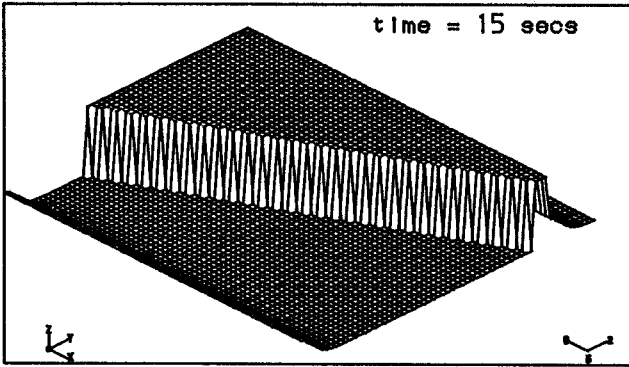


FIG 22 Total reflection of a solitary wave
 $d=10\text{m}$ $a=1\text{m}$ $DX=14\text{m}$ $DT=1\text{s}$ Wall at 60 deg

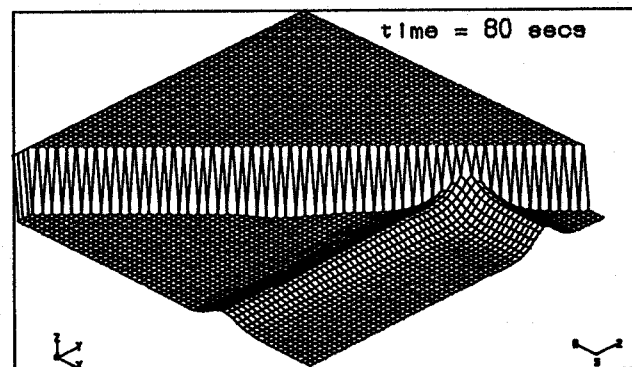
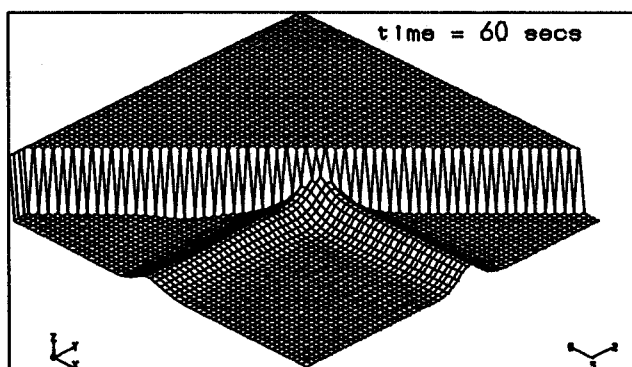
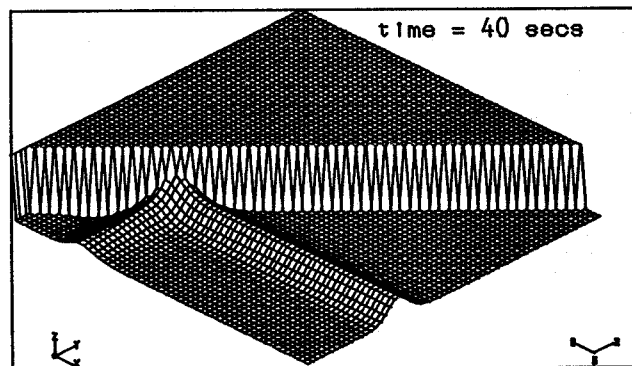
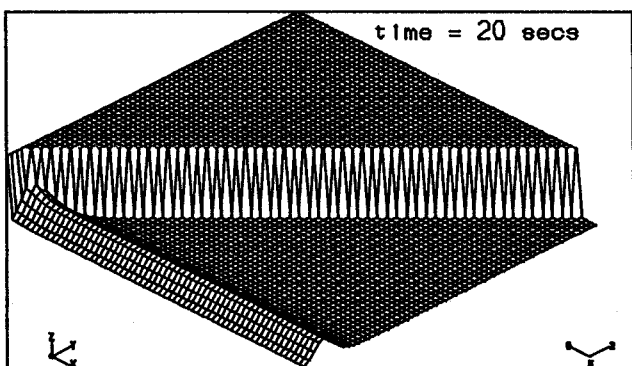


FIG 23 Partial reflection of a solitary wave
 $d=10\text{m}$ $a=1\text{m}$ $DX=14\text{m}$ $DT=1\text{s}$ $R = 0.5$

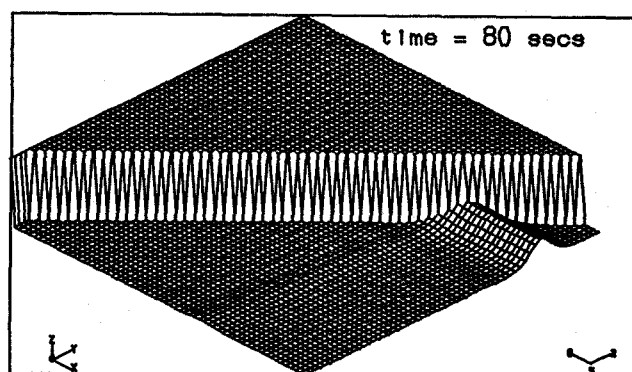
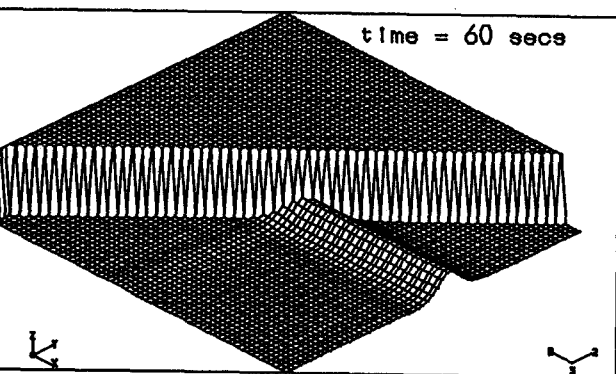
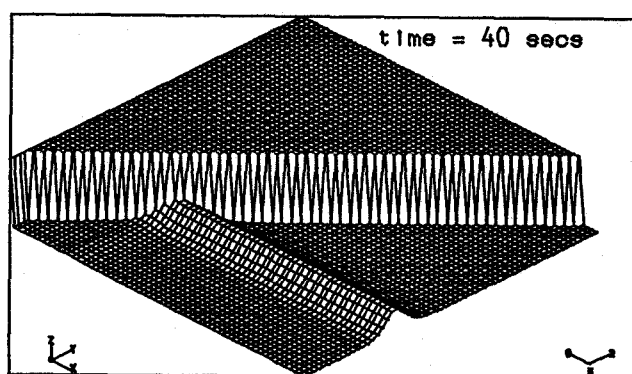
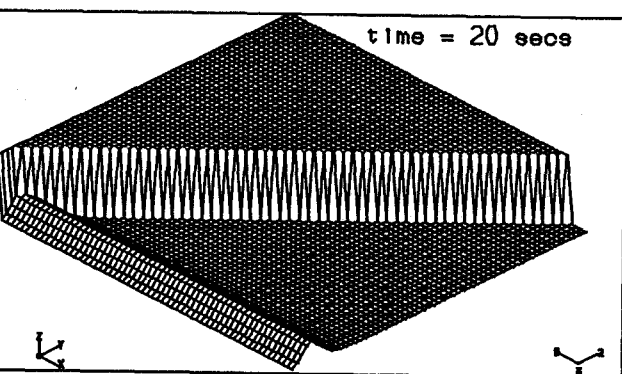


FIG 24 Absorption of a solitary wave
 $d=10\text{m}$ $a=1\text{m}$ $DX=14\text{m}$ $DT=1\text{s}$ Wall at 45 deg

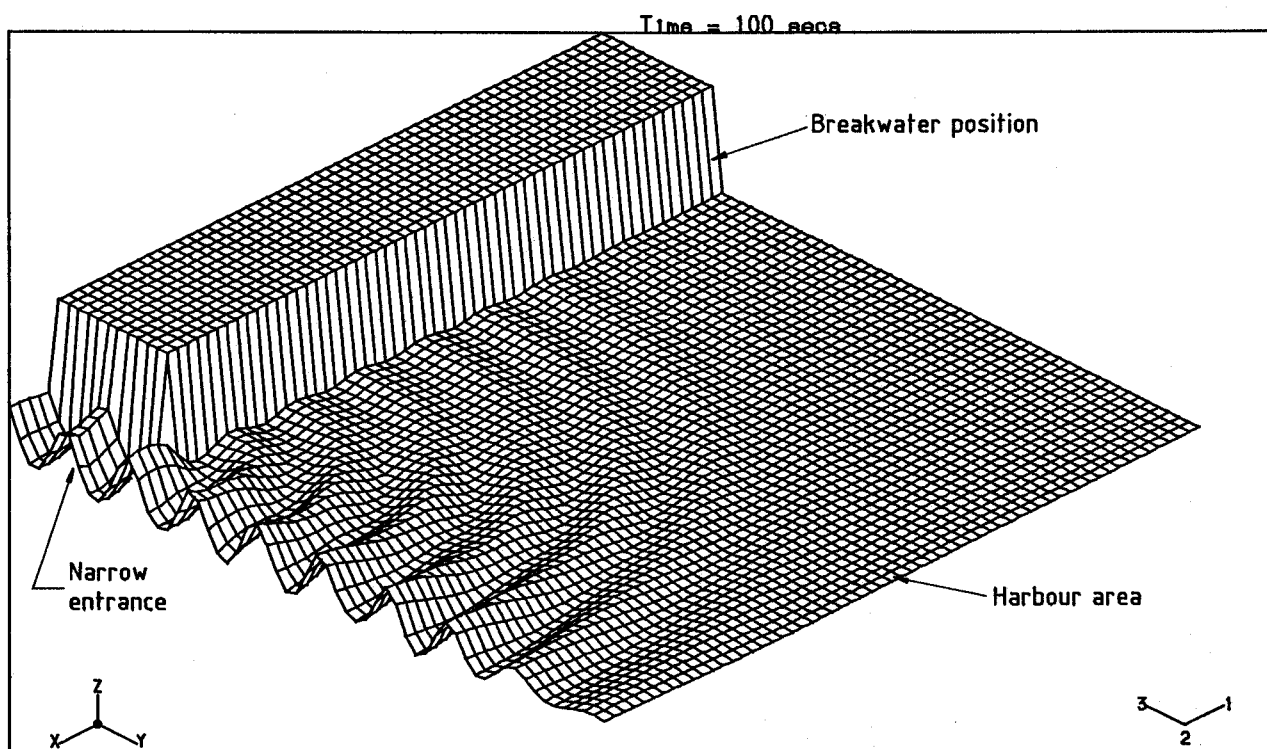


FIG 25 Non-linear diffraction --Free surface
 $d=10\text{m}$, $a=0.5\text{m}$, $DX=10\text{m}$, $DT=1\text{ sec}$

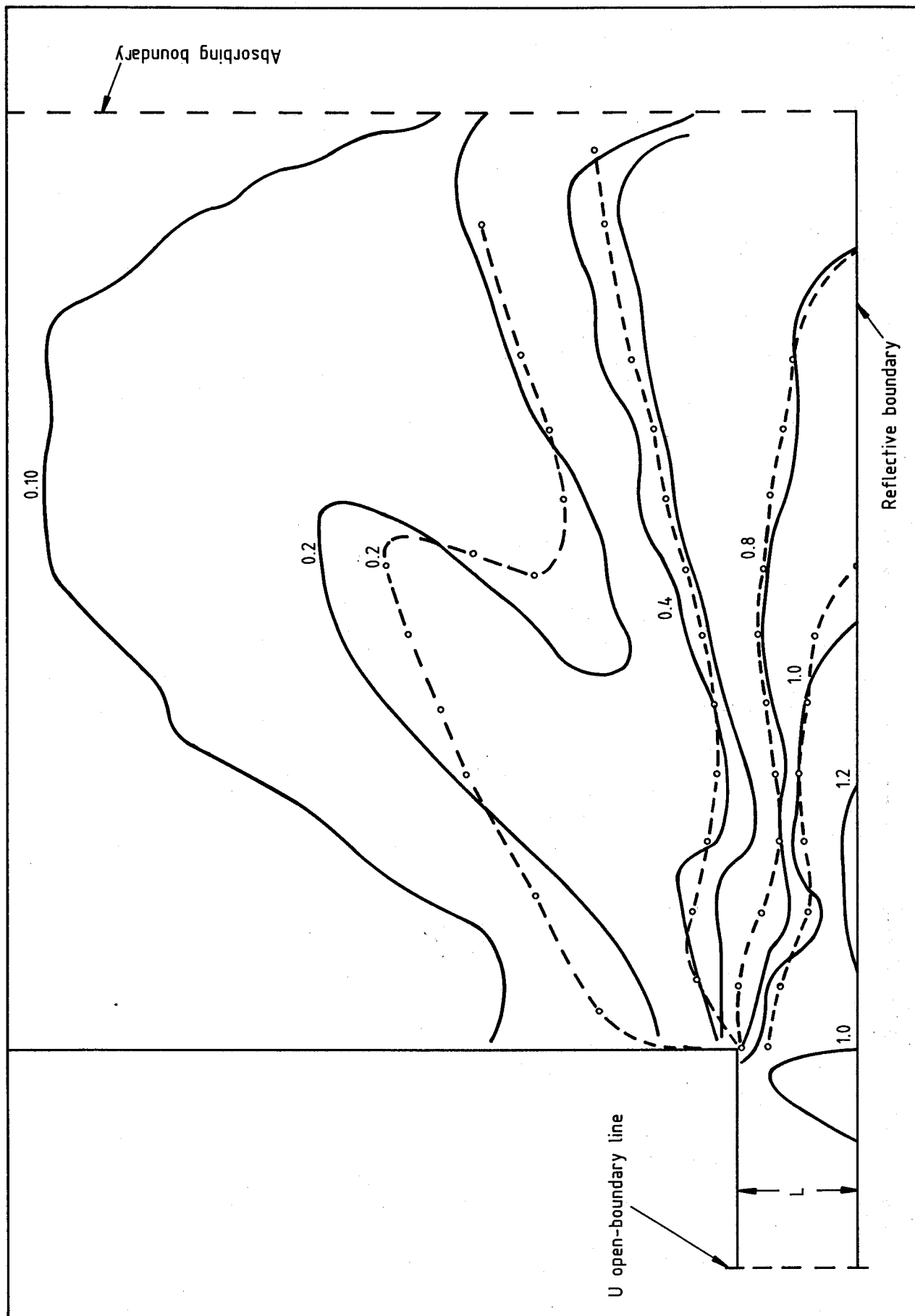


Fig 26 Comparison between theory (---) and model (—) - linear diffraction

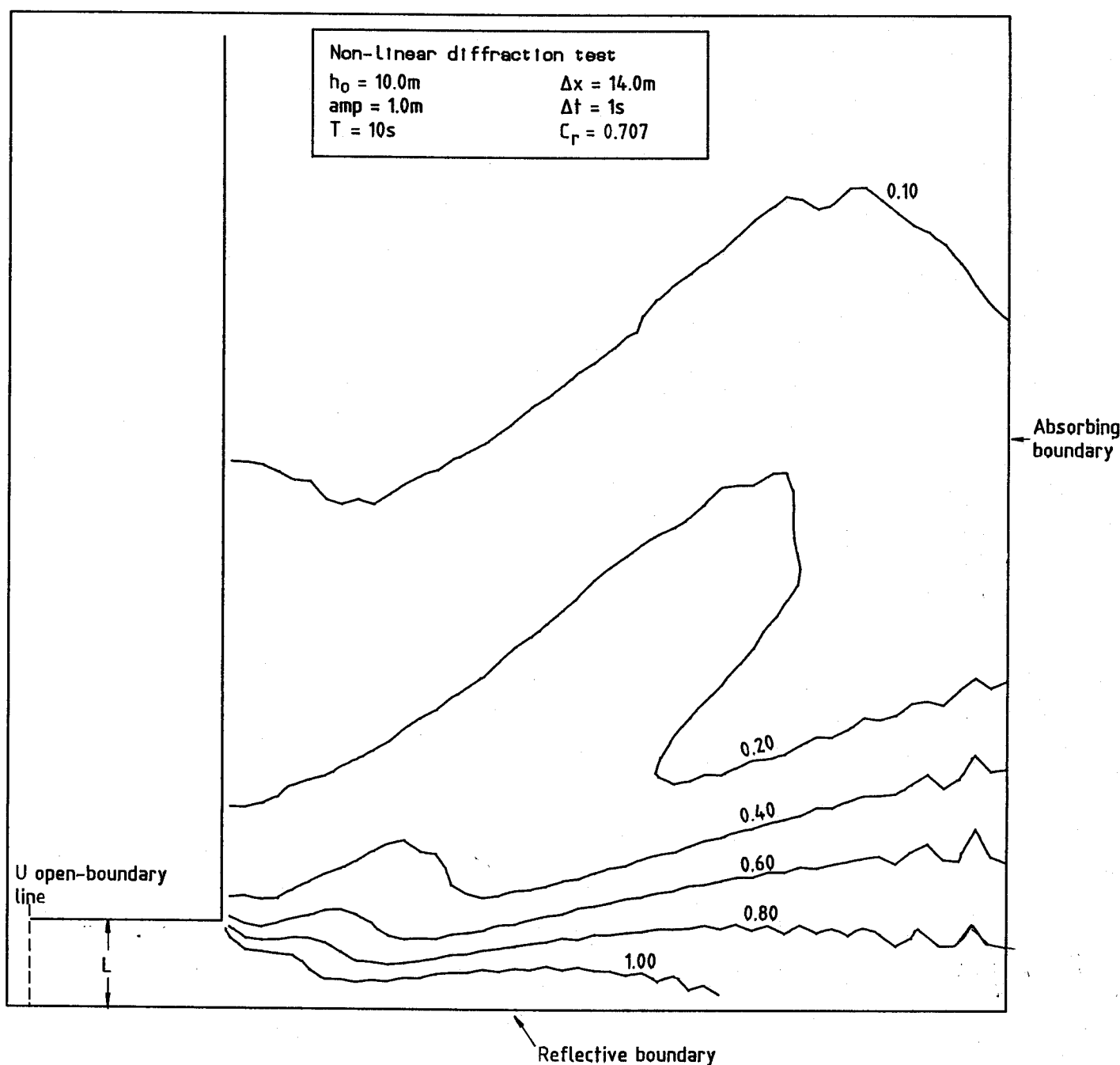


FIG 27 Diffraction coefficient contours - Sinusoidal incident wave

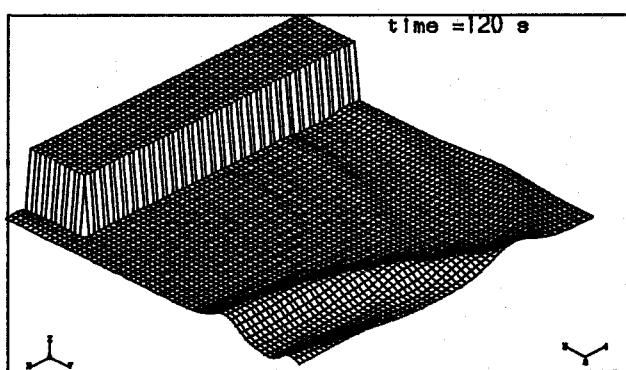
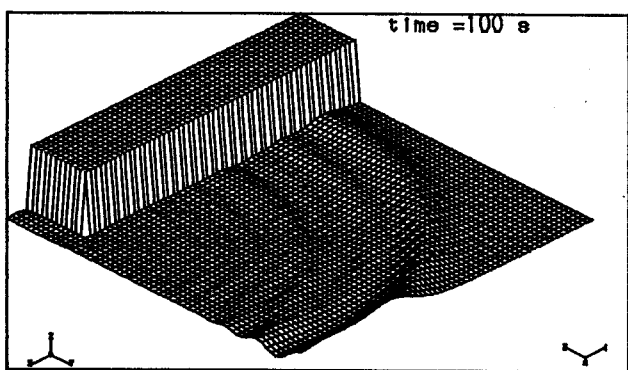
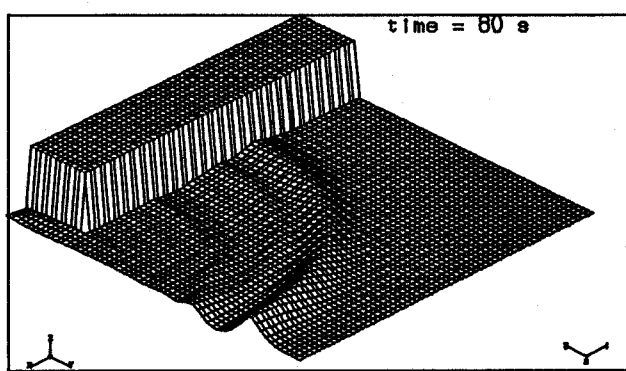
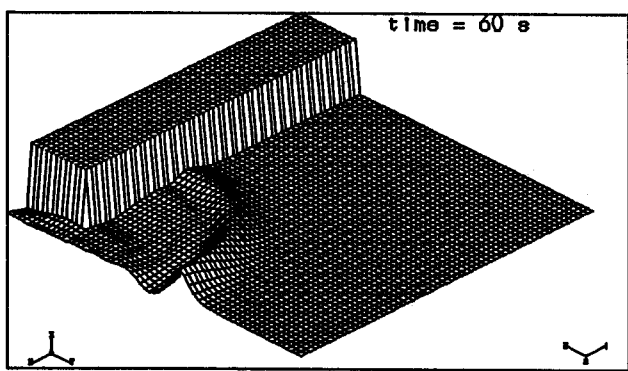
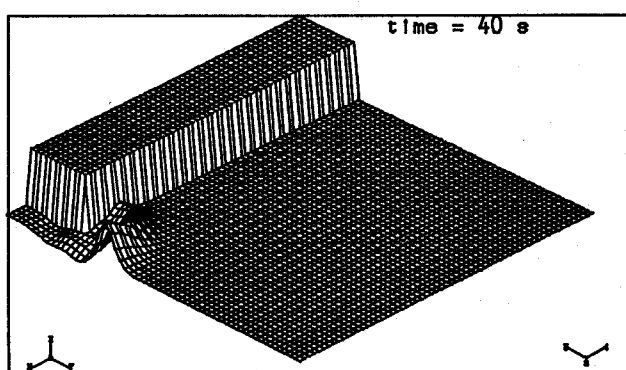
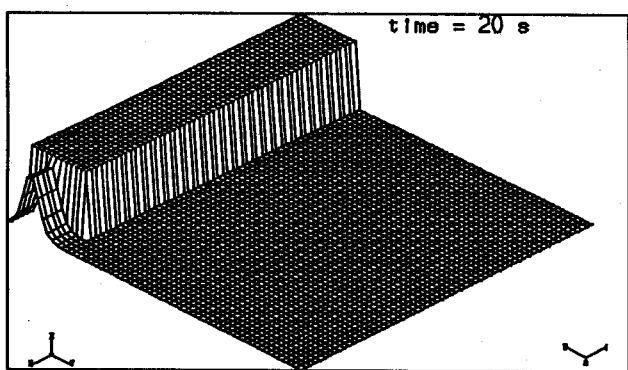


FIG 28 Diffraction/Reflection of solitary wave
in a closed basin

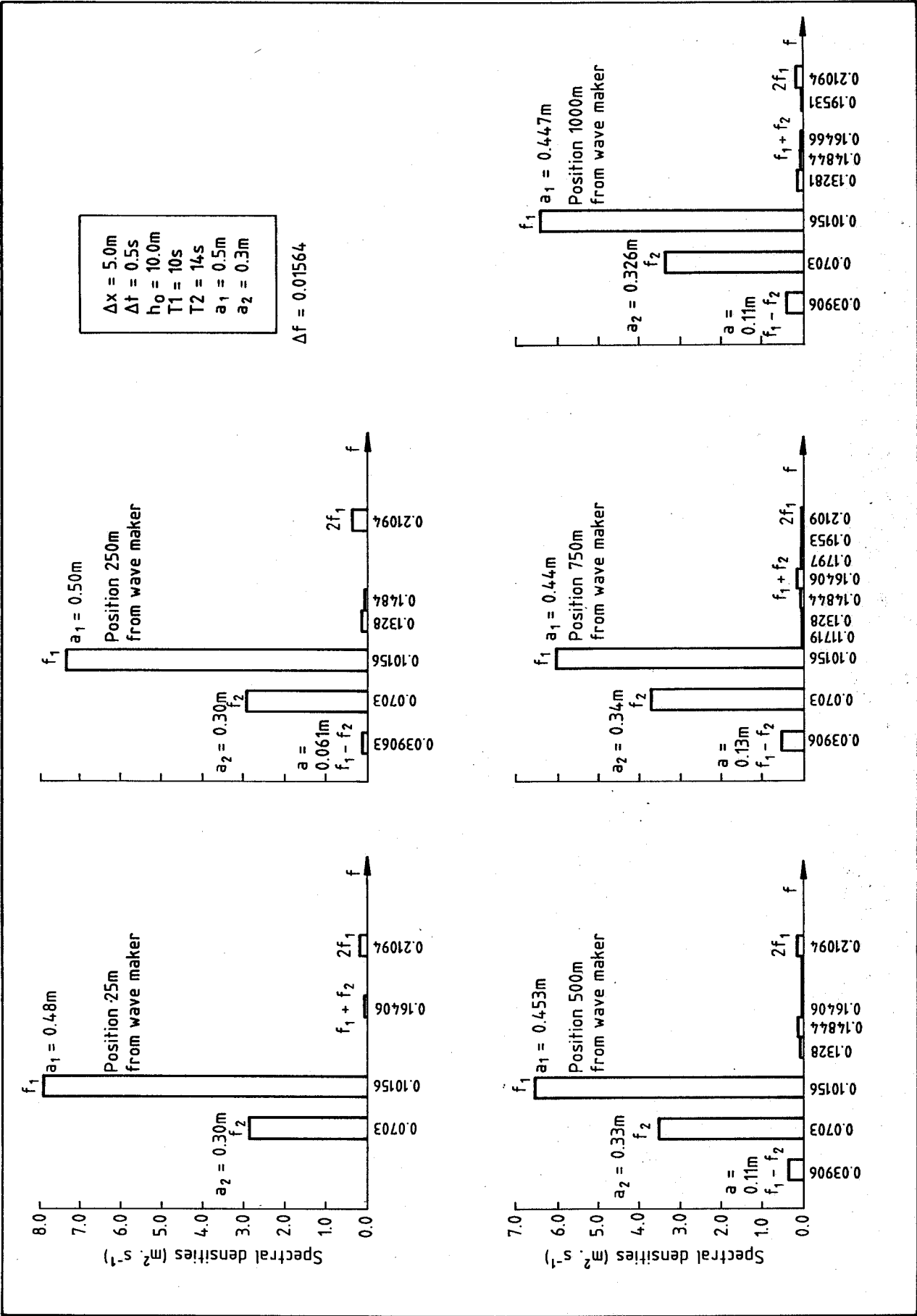


Fig 29 Spectra along the flume - No set-down wave at the open boundary

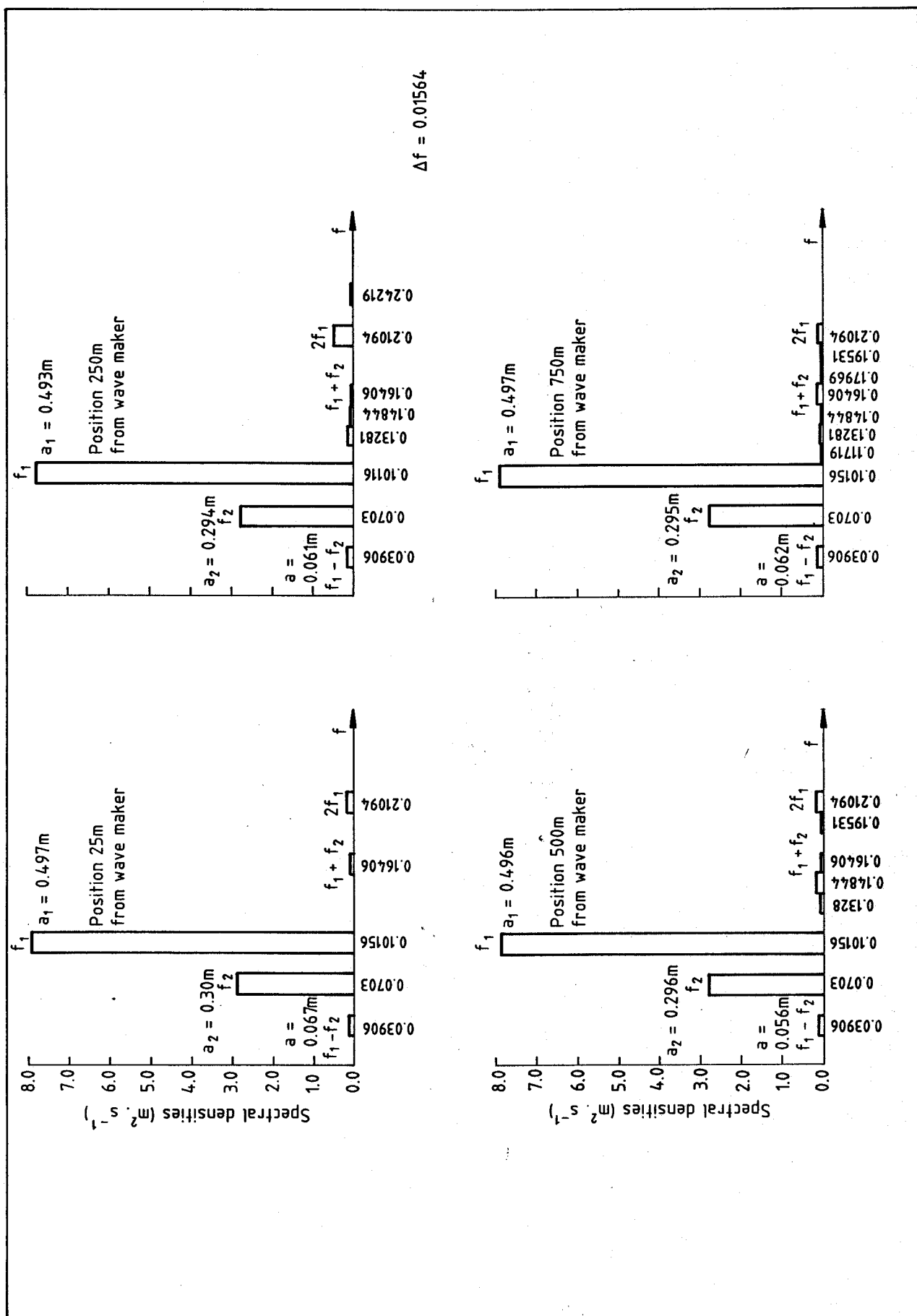


Fig 30 Spectra along the flume - Set-down wave is added at the open boundary

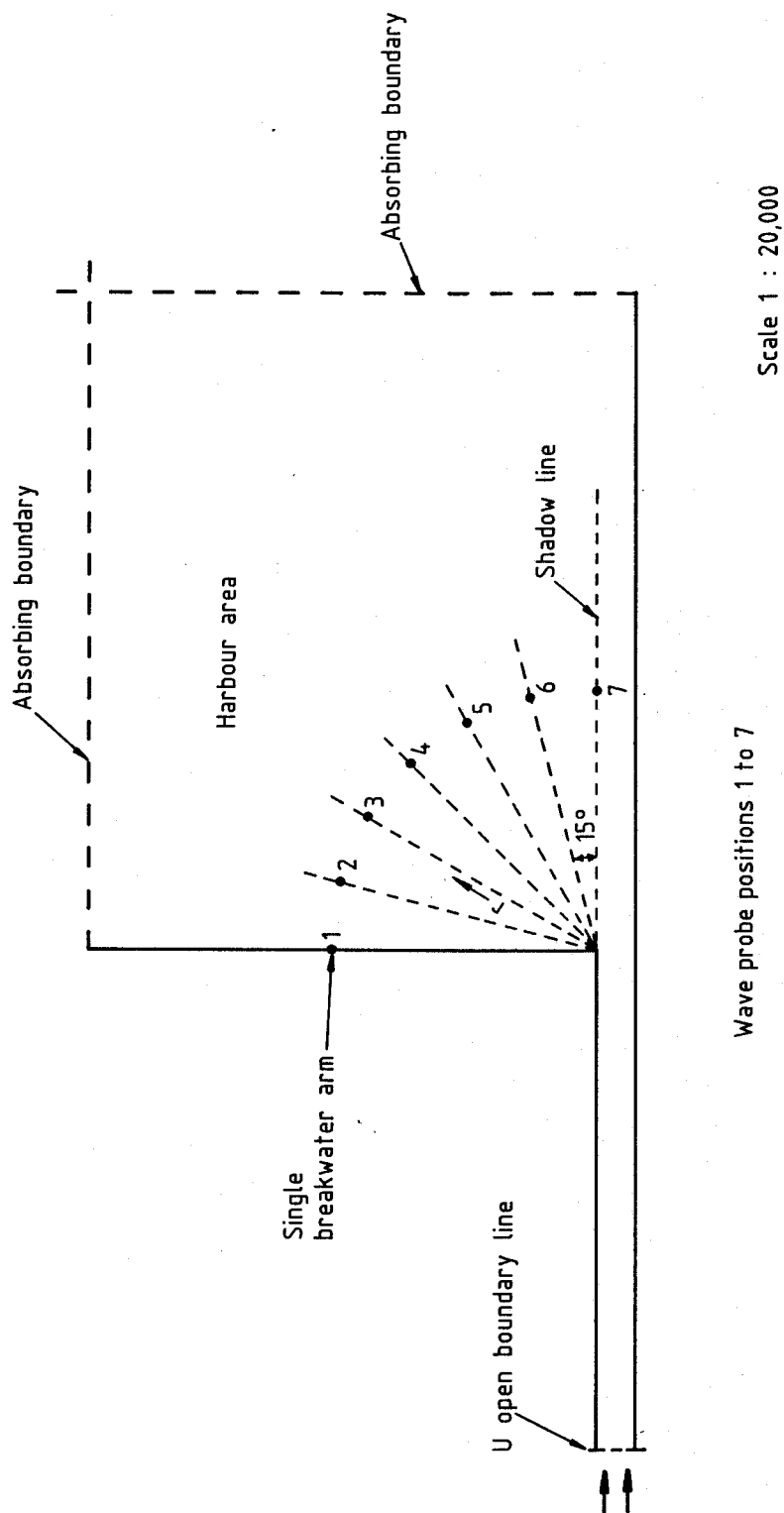
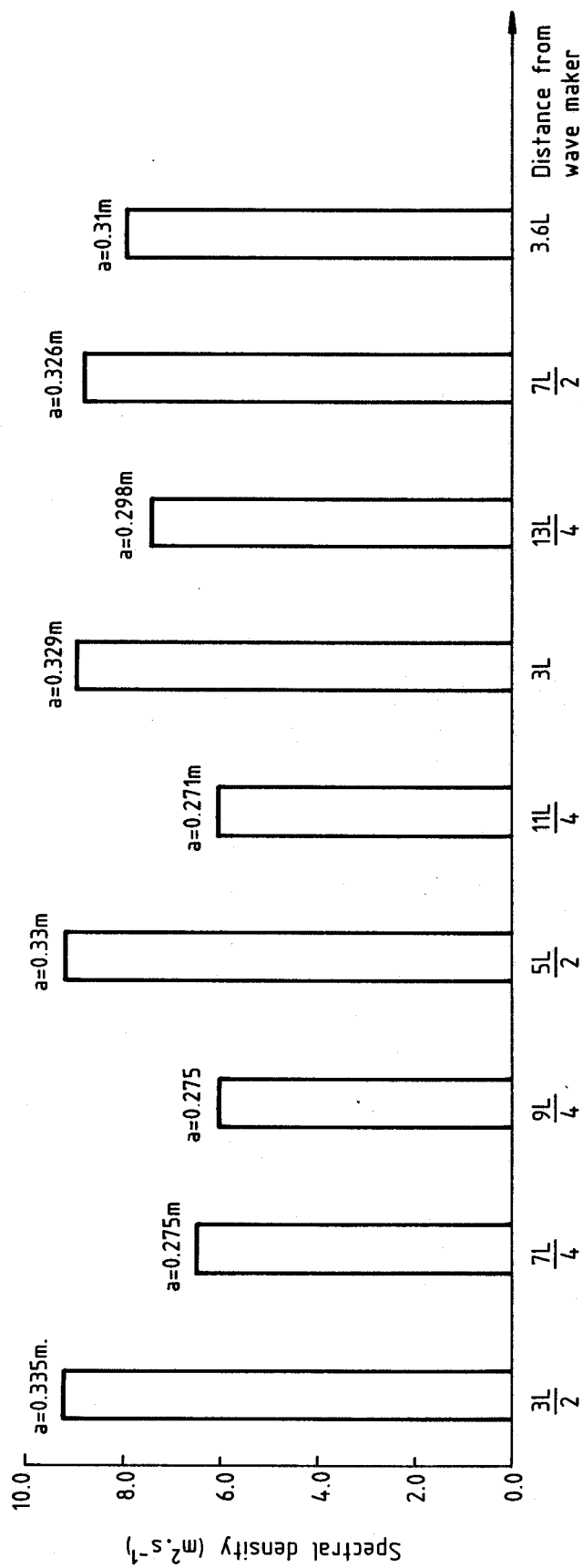


Fig 31 Experimental layout



$f = 0.03$
 $\Delta f = 0.005964$
 $L = 513.33m$

Reflection coef. $\lambda = 0.09$
 Phase $\beta = 315^\circ$

Fig 32 Spectral densities - free wave

Grid point 20

$-\Delta x-$

Open boundary line	-0.29	-0.32	-0.28	-0.24	-0.15	-0.35	0.05	0.15	0.23	0.28	0.32	0.31	0.25	0.16	0.03	-0.09	-0.21	-0.28	-0.32	-0.37
	-0.29	-0.32	-0.28	-0.24	-0.15	-0.35	0.06	0.15	0.23	0.28	0.32	0.31	0.25	0.16	0.03	-0.09	-0.21	-0.28	-0.32	-0.37
	-0.29	-0.32	-0.28	-0.24	-0.15	-0.35	0.06	0.15	0.23	0.28	0.32	0.31	0.25	0.16	0.03	-0.09	-0.21	-0.28	-0.32	-0.37
	-0.30	-0.32	-0.28	-0.24	-0.15	-0.35	0.06	0.15	0.23	0.28	0.32	0.31	0.25	0.16	0.03	-0.09	-0.21	-0.28	-0.32	-0.37
	-0.30	-0.32	-0.28	-0.24	-0.15	-0.35	0.06	0.15	0.23	0.28	0.32	0.31	0.25	0.16	0.03	-0.09	-0.21	-0.28	-0.32	-0.37
	-0.30	-0.32	-0.28	-0.24	-0.15	-0.35	0.06	0.15	0.23	0.28	0.32	0.31	0.25	0.16	0.03	-0.09	-0.21	-0.28	-0.32	-0.37

Point 41

Point 21

	-0.26	-0.23	-0.11	-0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02
	-0.26	-0.23	-0.11	-0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02
	-0.26	-0.23	-0.11	-0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02
	-0.26	-0.23	-0.11	-0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02
	-0.26	-0.23	-0.11	-0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02
	-0.26	-0.23	-0.11	-0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02

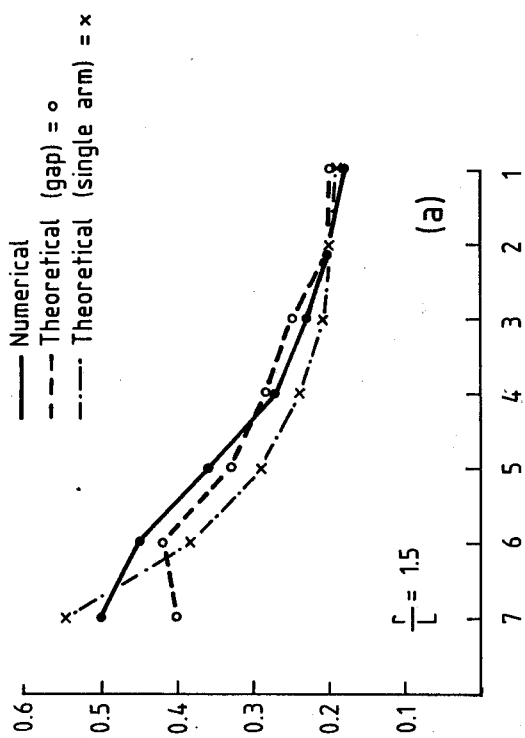
Point 42

Breakwater tip

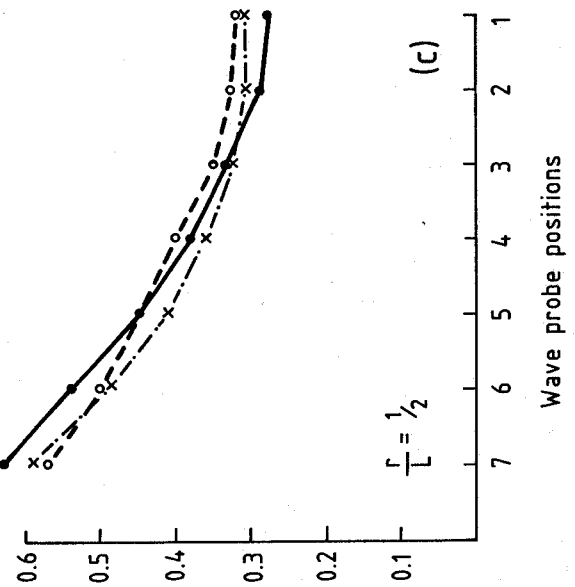
	0.22	0.19	0.23	0.23	0.23	0.23	0.23	0.23	0.23	0.23	0.23	0.23	0.23	0.23	0.23	0.23	0.23	0.23	0.23	0.23
	0.22	0.19	0.23	0.23	0.23	0.23	0.23	0.23	0.23	0.23	0.23	0.23	0.23	0.23	0.23	0.23	0.23	0.23	0.23	0.23
	0.22	0.19	0.23	0.23	0.23	0.23	0.23	0.23	0.23	0.23	0.23	0.23	0.23	0.23	0.23	0.23	0.23	0.23	0.23	0.23
	0.22	0.19	0.23	0.23	0.23	0.23	0.23	0.23	0.23	0.23	0.23	0.23	0.23	0.23	0.23	0.23	0.23	0.23	0.23	0.23
	0.22	0.19	0.23	0.23	0.23	0.23	0.23	0.23	0.23	0.23	0.23	0.23	0.23	0.23	0.23	0.23	0.23	0.23	0.23	0.23
	0.22	0.19	0.23	0.23	0.23	0.23	0.23	0.23	0.23	0.23	0.23	0.23	0.23	0.23	0.23	0.23	0.23	0.23	0.23	0.23

$\Delta x = 29.0m$

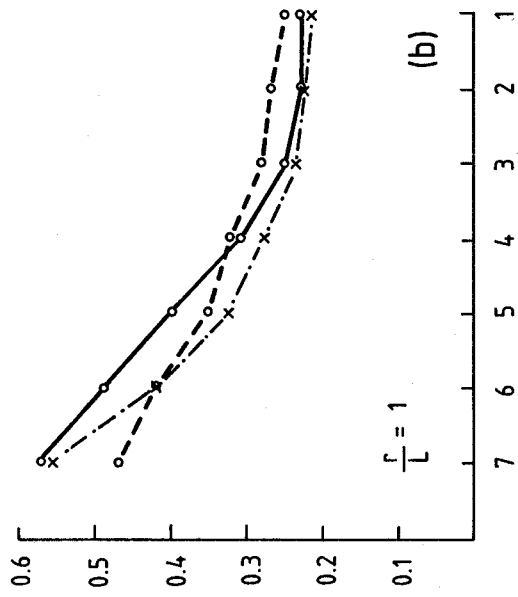
Fig. 33 Surface elevations (m) - ITER = 800



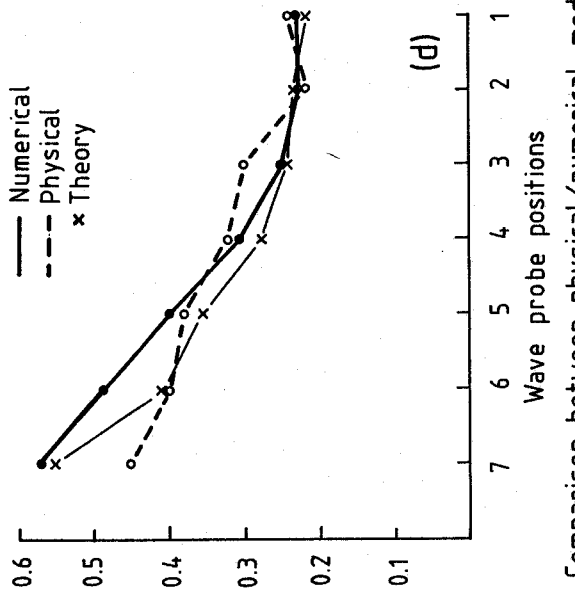
(a)



(c)



(b)



(d)

Comparison between physical/numerical model

Fig 34 Diffraction coefficients

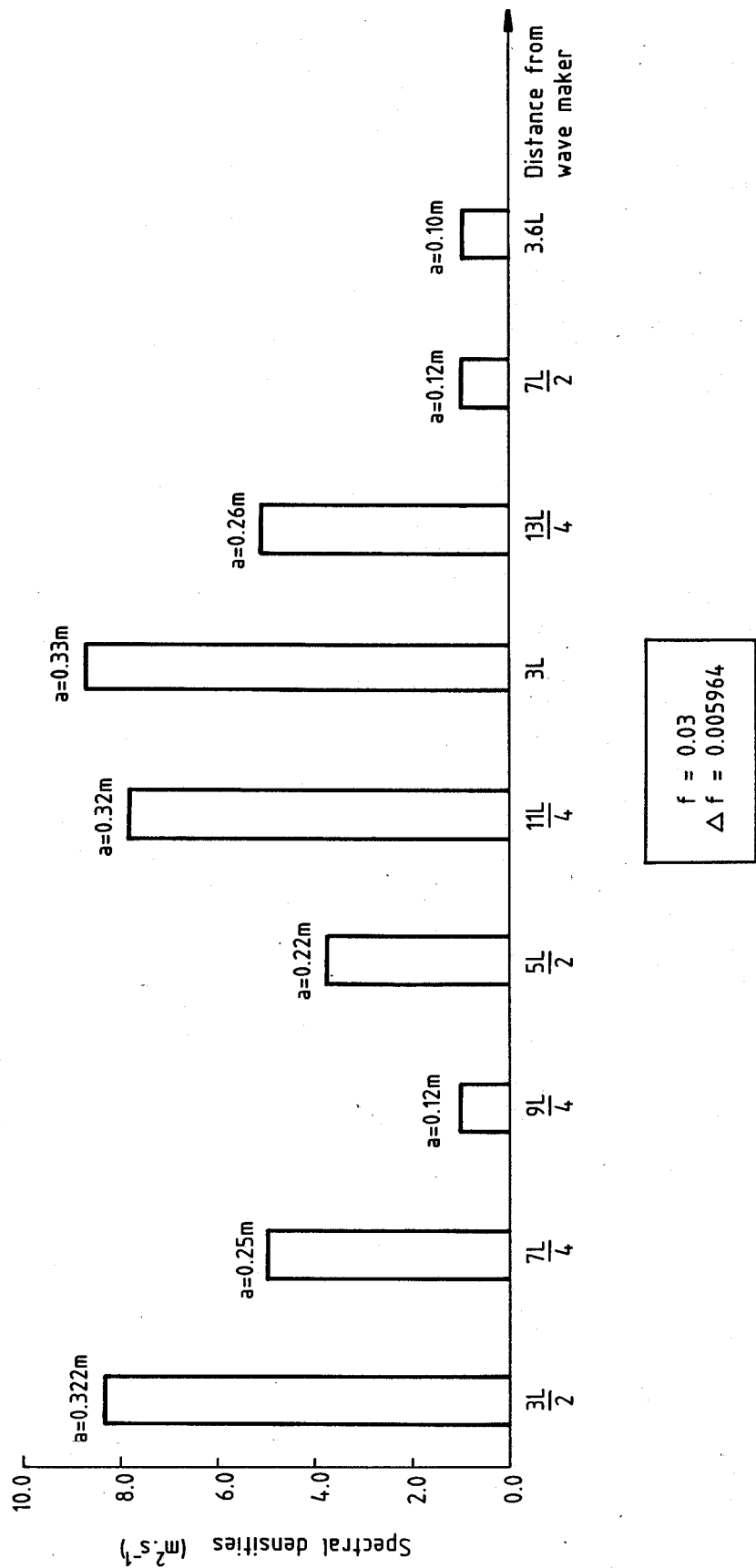


Fig 35 Spectral densities - set down wave

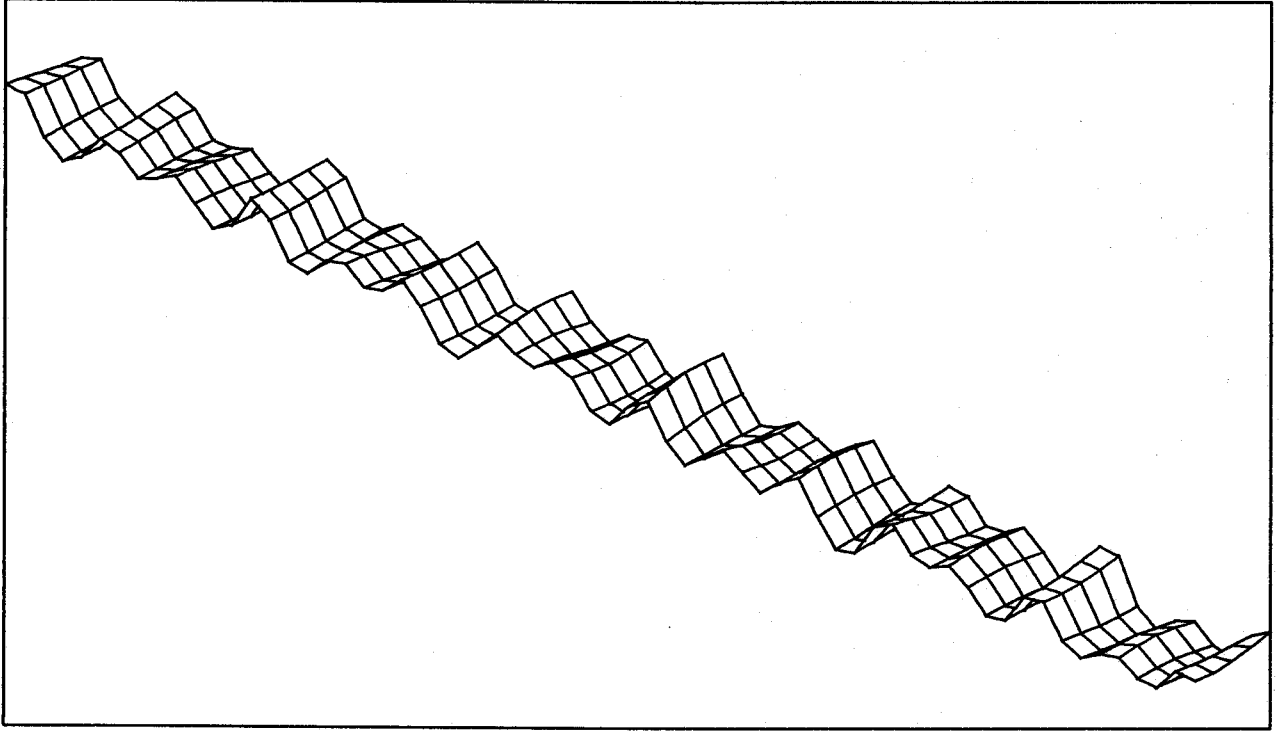


FIG 36 Diffraction of set-down wave
Breakwater flume ITER = 800

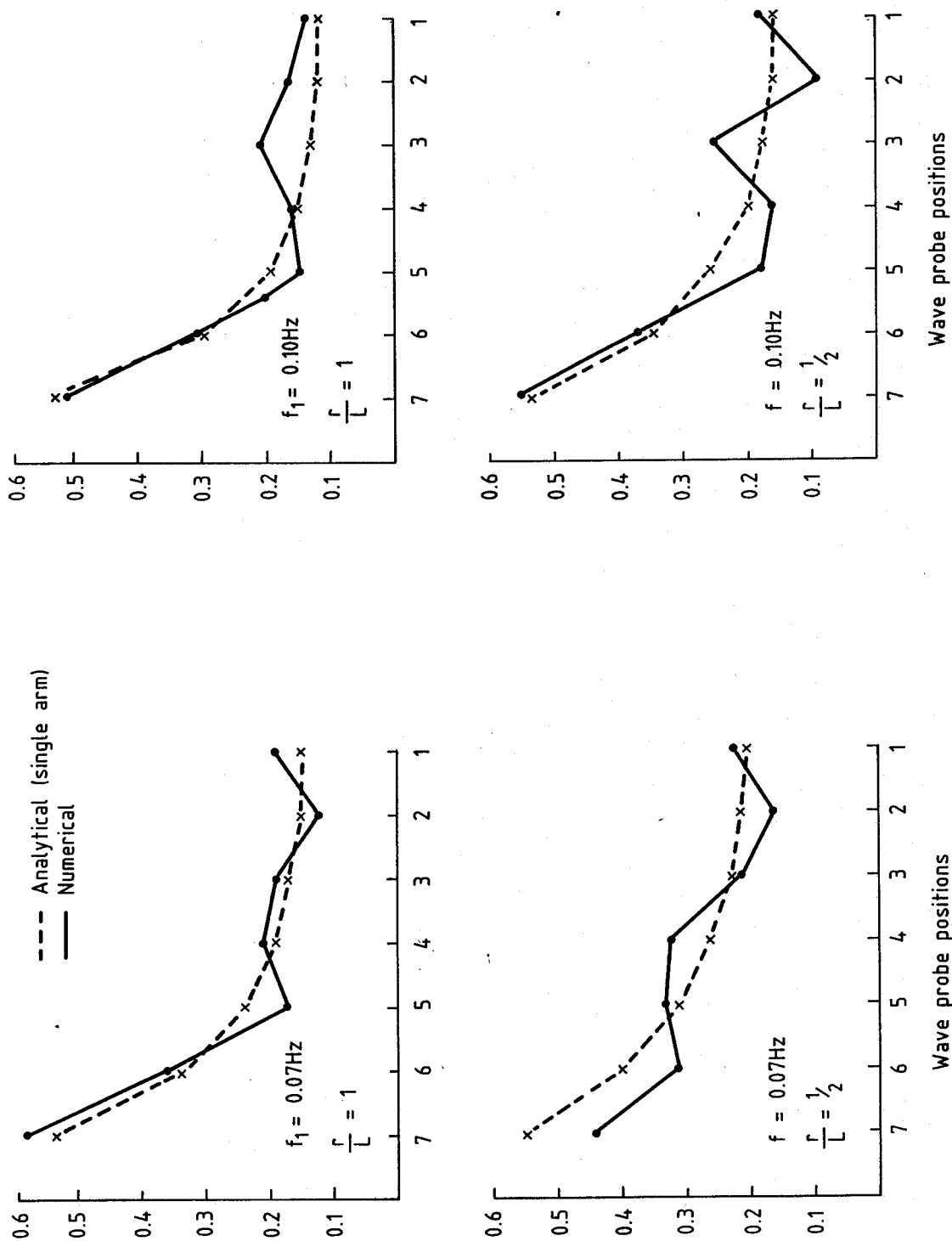


Fig 37 Diffraction of primary waves

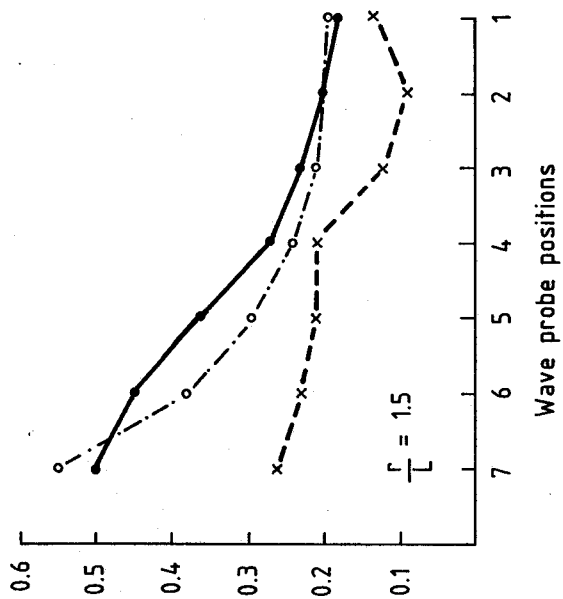
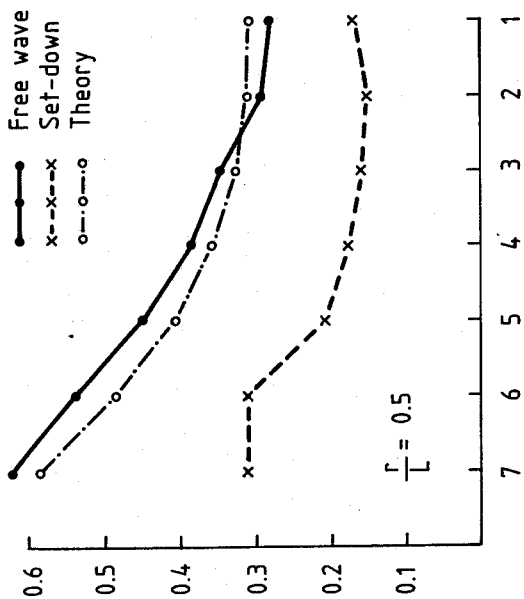
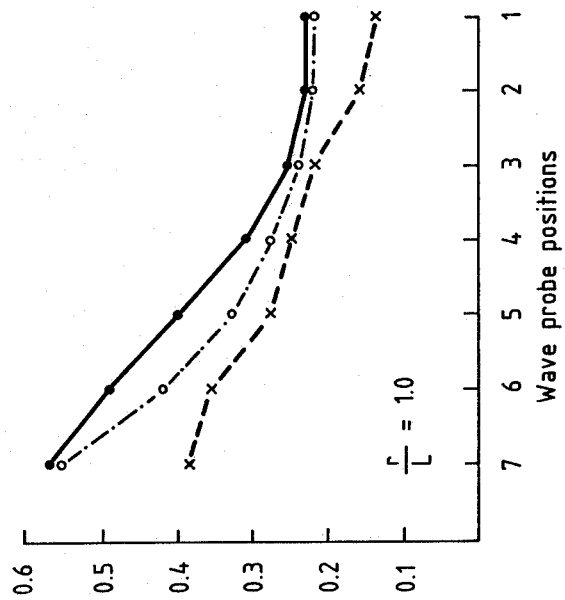
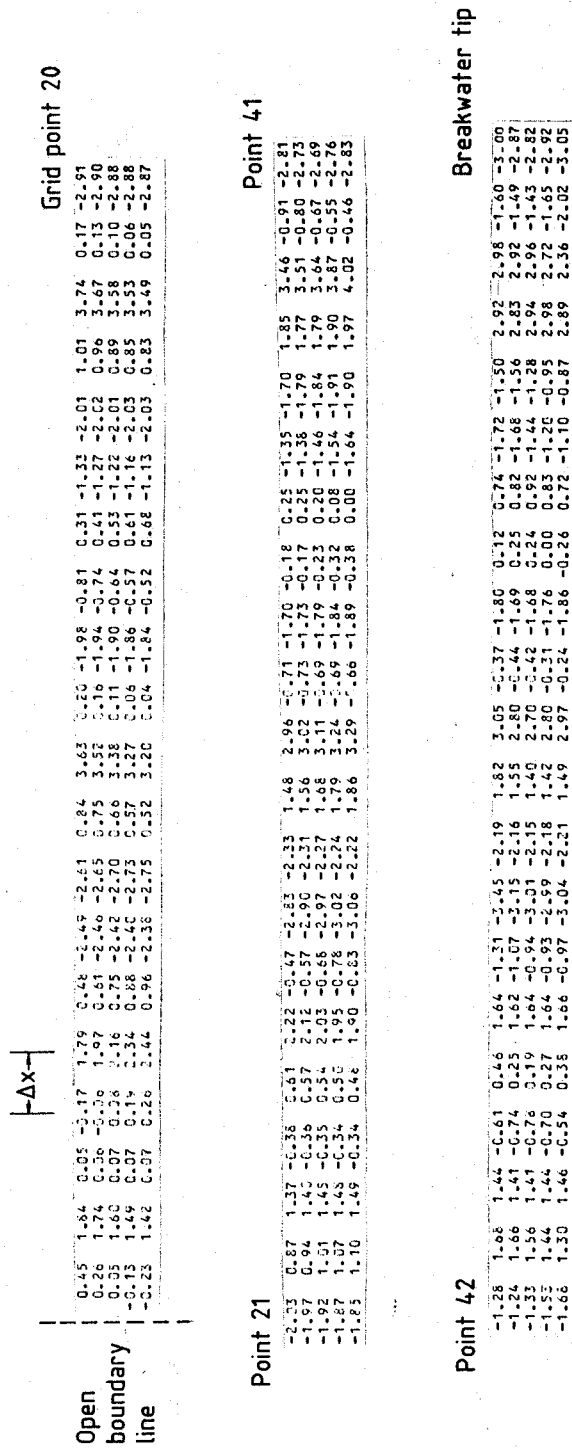


Fig 38 Diffraction of set-down wave

Fig. 39 Surface elevations (m) - ITER = 300



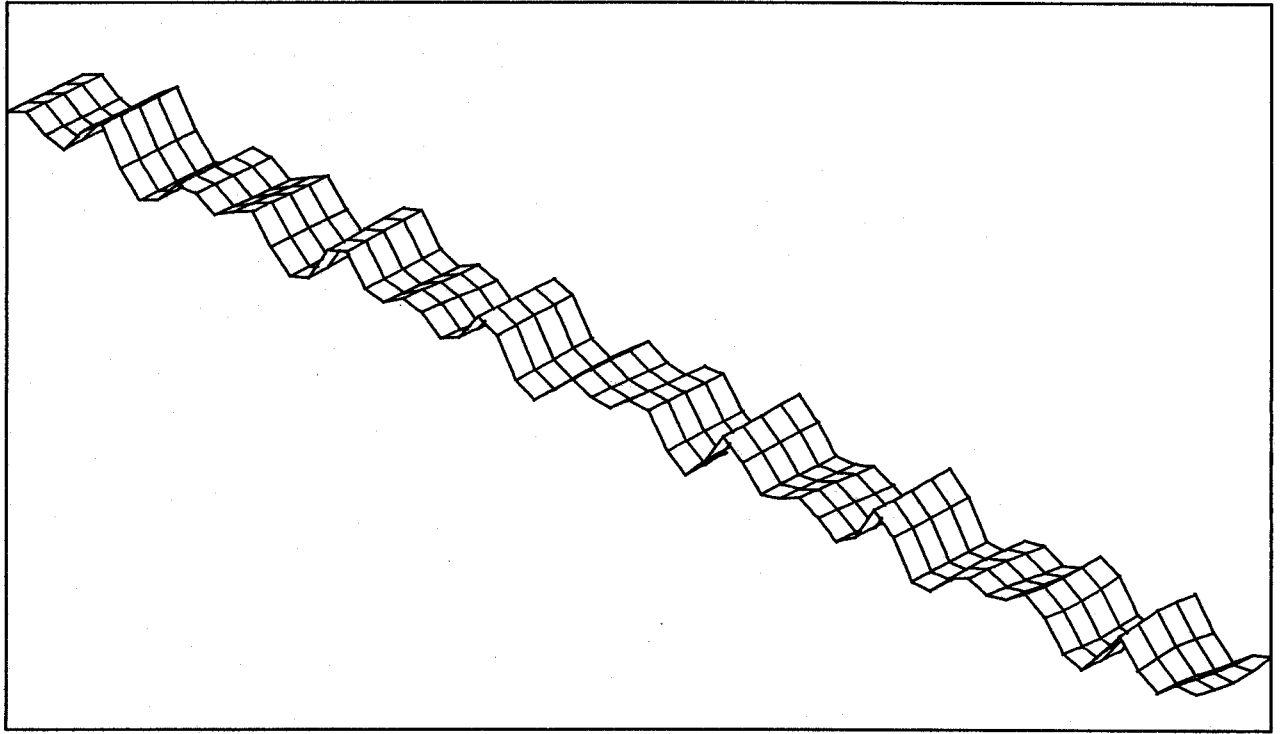


FIG 40 Diffraction of set-down wave
Breakwater flume ITER = 300

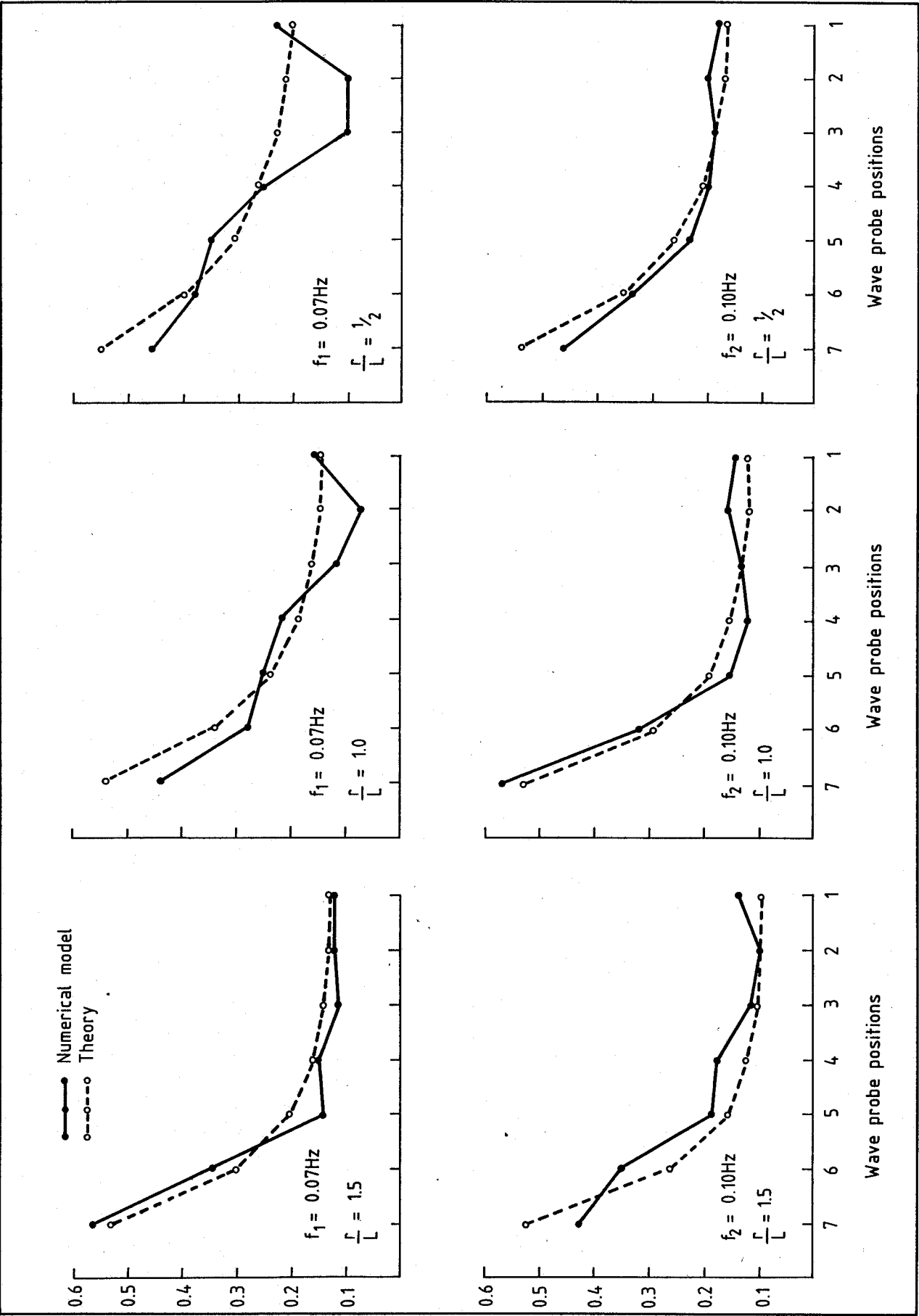


Fig 41 Diffraction of primary waves - enlarged model

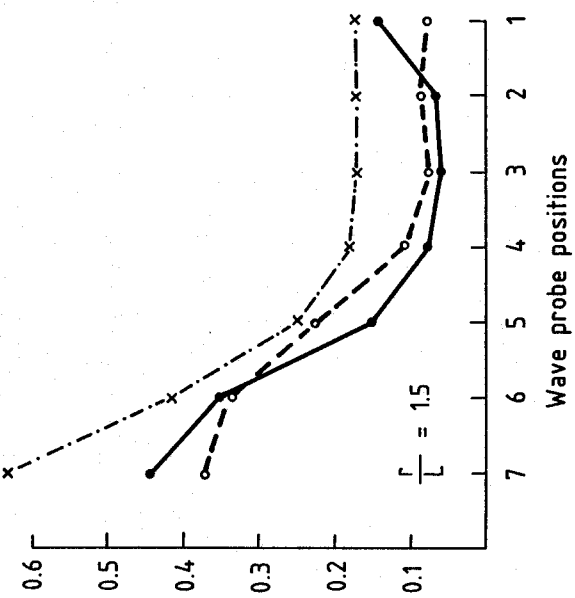
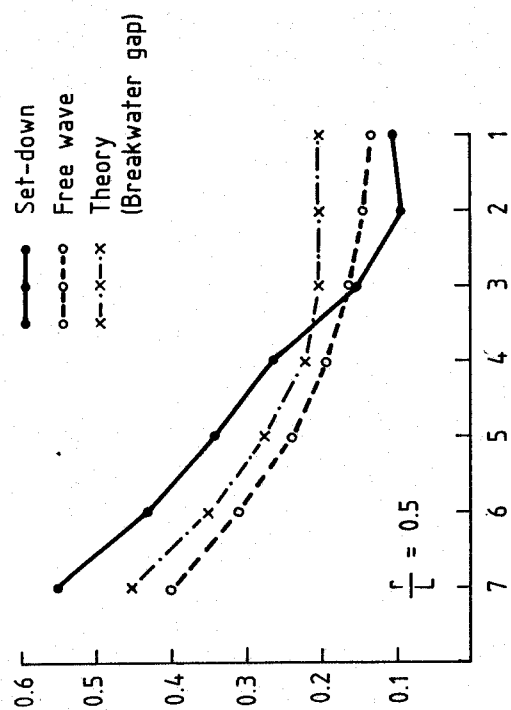
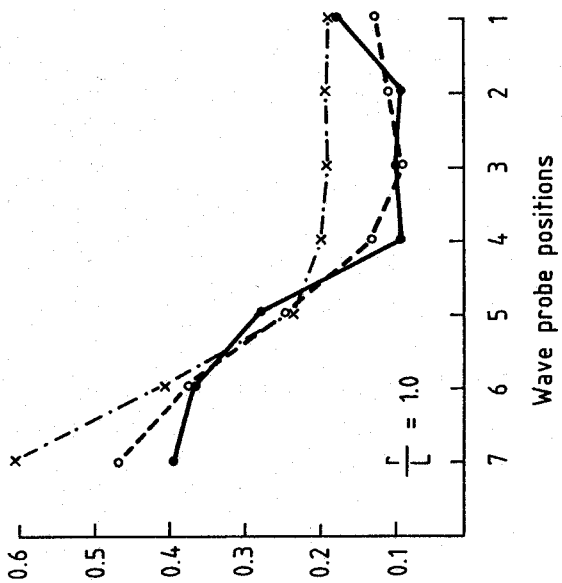


Fig 42 Diffraction of set-down - enlarged model